Assignment for Lecture 3

STATISTICAL DISTRIBUTIONS, BAYES' THEOREM,
AND GAUSSIAN RANDOM VARIABLE GENERATION

Lecture Date: 3/17/2026

"C" denotes for "computational" problems, language suggestion: Python/Julia

please include codes and results with analyses for computational problems

please write in pdf format and submit to `bjcai@fudan.edu.cn` ***before the lecture of 3/24/2026***

1. Prove Jensen's inequality by mathematical induction. Similarly, prove it by using the Taylor expansion of a function $f(x)$ to second order, $f(x) \approx f(\mu) + f'(\mu)(x-\mu) + 2^{-1} f''(\mu)(x-\mu)^2 \geq f(\mu) + f'(\mu)(x-\mu)$, for a convex function.

2. Prove the relation $\mathrm{kurt}[x] \geq \mathrm{skew}^2[x] - 2$ for any distribution.

3. If x and y are independent with each other, prove $\mathscr{M}_{x+y}(t) = \mathscr{M}_x(t)\mathscr{M}_y(t)$.

4. [C] Use the exponential distribution to verify the central limit theorem.

5. [C] Radioactive decay is an important problem in nuclear physics. Consider a sample containing N nuclei that decay at a rate $\lambda\,\mathrm{s}^{-1}$. The physics of the process specifies that the decay rate is given by $\mathrm{d}N/\mathrm{d}t = -\lambda N$, where the nuclei that decay during the time interval $\mathrm{d}t$ can be selected randomly. The solution of the decay equation is $N = N_0 e^{-\lambda t}$, where N_0 is the initial number of nuclei and λ is related to the "half-life" of the system. Given a sample containing 10^5 radioactive nuclei, each of which decays at rate q per second, what is the half-life of the sample if $q = 0.15$?