SHORT LECTURES ON
MATHEMATICS/PHYSICS/ALGORITHM
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$\mathbf{f/f_0 = 1 + \cdots}$

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In this lecture, we introduce the fundamental concepts of perturbative calculations. The basic method for doing perturbative calculations is the Taylor's expansion of a function, based on which one can obtain the perturbative form of the relevant quantity as $f/f_0 = 1+\cdots$ where f_0 is the leading/main order contribution. Elementary examples on using perturbative calculations are given in the work. Moreover, a few important structures/properties encapsulated in the perturbative calculations will be discussed in details, such as the duality, harmonic oscillation and spontaneous symmetry breaking. The concept of effective theories is closely related with the perturbative calculations, and which is also discussed via considering the nonlinear effects introduced into the harmonic system.

I. INTRODUCTORY COMMENTS

Determination/calculation of a physical quantity is a fundamental issue in physics, which is also important for other subjects. However, it is often very difficult to obtain the quantity analytically, and in this sense the perturbative methods are frequently adopted. Generally, one assumes that the perturbative form of a quantity f as $f = f_0+f_1+f_2+\cdots$, where the f_0 is called the leading term, which contributes most to f. On the other hand, $f_1+f_2+\cdots$ is as a whole called the correction and is often a small quantity in the sense that,

$$\left|\frac{f_j}{f_0}\right| \ll 1,\ \frac{\sum_j |f_j|}{|f_0|} \ll 1,\ \left|\frac{f_{j+1}}{f_j}\right| < 1,\ \lim_{j\to\infty}\frac{f_{j+1}}{f_j} = 0, \tag{1.1}$$

where $j = 1,2,\cdots$. Moreover, we could write the f in the perturbative form as,

$$f = f_0(1+F_1+F_2+\cdots),\ F_j = f_j/f_0. \tag{1.2}$$

The basic aim of any forms of the perturbative calculation is to obtain the expressions of $F_1,F_2,\cdots$ and consequently estimate their effects to the leading term f_0 (or "1" compared with $F_1,F_2,\cdots$).

The quantity f is assumed to be a function of a few parameters involved namely $f = f\{\phi_i\} = f(\phi_1,\cdots,\phi_n)$, among which some parameters ϕ_α are reasonable to be used to construct the perturbative schemes, e.g., $f\{\phi_i\} = f_{0,\alpha}(1 + g_{1,\alpha}\phi_\alpha + g_{2,\alpha}\phi_\alpha^2 + \cdots)$, with $g_{1,\alpha},g_{2,\alpha},\cdots$, the expansion coefficients. One could effectively define the convergence factor as $\Phi_{j,\alpha} = g_{j+1,\alpha}/g_{j,\alpha}$, which fundamentally fulfills that $\lim_{j\to\infty}\Phi_{j,\alpha} = 0$. In many problems, the factor $\Phi_{j,\alpha}$ approaches to zero very quickly as shown in Fig. 1. In this figure, the pattern b) is reasonable while patterns a) and c) are not appropriate for doing perturbative calculations. In situations with multiple parameters involving the perturbative calculation, we often take perturbative calculations on one of them while fixing the others, and after that the number of the perturbative parameters is reduced by one and consequently we could do relevant calculations on the corresponding coefficients.

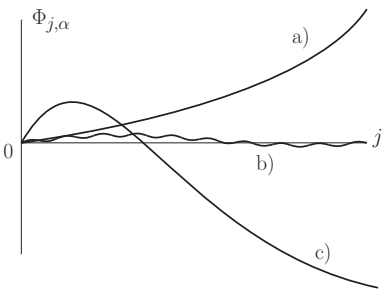

Fig. 1: Convergence of the perturbative calculations, where pattern b) is reasonable while patterns a) and c) are not.

In some problems, the main (leading) contribution "1" in (1.2) does not exist, i.e., the perturbation starts with the small quantity. in addition, the even more general form of the perturbative calculation takes the one $f\{\phi_i\} = f_{0,\alpha}[1+\sum_{j=1} g_{j,\alpha}\phi_\alpha^{\rho_j}]$, where the constant "1" here could also be zero (all the other non-zero constants could always be transformed into "1"). There may exist singular terms in the final expressions of the perturbative calculations, e.g., the terms like $(\phi_\alpha)^\epsilon \log\phi_\alpha$, and these terms are often originated from the singular properties encapsulated in the problems themselves. Furthermore, the indexes ρ_j's are determined by the structure and/or symmetry of the problem involved. For example, in the nonlinear effects of the simple pendulum to be explored in the following part the indexes ρ_j's take even values from 2 due to the left-right symmetry of the system. It could be generally stated that the perturbative structure of the calculation reflects the complexity of the series expansion from the viewpoint of mathematics.

If the exact form of a quantity W is wrote as

$$W = \omega\times 10^\delta, \tag{1.3}$$

the aim of order of magnitude estimation and perturbative calculations is to roughly give the value of the index δ while that of the index ω needs the whole/complete theories. Estimating the δ is already very important (more generally the determination of the sign of W has very useful consequences in some problems).

The lecture is organized as follows: In section II we give an elementary example namely using perturbative calculations to determine the root of a simple algebraic equation with order beyond four. Some basic structures like the duality of the perturbative calculations are introduced within this example. Section III is devoted to the period of the simple pendulum system, where the nonlinear effects due to the finite swing angle $\chi_{\max}$ are calculated, then in section IV we generally discuss how to explore the nonlinear effects on the period of the system. In section V, we introduce the concept of spontaneous symmetry breaking which is essentially related with the perturbative calculations for some certain quantities. Section VI introduces another important form of the perturbative calculations namely the integration form, for problems one can not solve the quantity directly and instead one needs to compare both sides of the equation determining the quantity involved. In the appendix, we discuss the role played by the polynomial in the perturbative calculations. Along the development of the lecture, more words are given using examples instead of formal theories. Different types of exercise are provided in the context for further familiarizing the methods developed.

II. EXAMPLE: SOLUTION OF AN ALGEBRAIC EQUATION

The main feature of the estimation on the order of magnitude and approximated perturbative calculations could be clearly demonstrated in some very elementary mathematical problems. For instance, it could be shown when trying to solve the following simple algebraic equation,

$$x^n(t) = \Omega + tx(t)/\Lambda,\ x(t)\in \mathrm{R}^+,\ n\in \mathrm{N}^+,\ n\geq 5,\ t\geq 0, \tag{2.1}$$

where t and Λ are two parameters (e.g., t is the time) with Λ a positive constant. As we all know there is no closed formula for the simple algebraic equation if the order of the equation is larger than or equal to four. In this sense we need use either numerical algorithms or approximated methods to investigate the root of the equation like (2.1).

If the "time" t is small near zero, the second term on the left hand side of Eq. (2.1) could be treated as a perturbation and in this case we could assume $x(t) \approx x_0[1+\delta(t)] = \Omega^{1/n}[1+\delta(t)]$ with $\delta(t)$ a small correction to the leading order solution "1". Expanding both sides of Eq. (2.1) to order $\delta(t)$, one obtains $\delta(t) = t\Omega^{1/n-1}/n\Lambda$ and then,

$$x(t) \approx \Omega^{1/n}\left(1+\frac{1}{n\Lambda}\Omega^{1/n-1}t\right). \tag{2.2}$$

The above (approximated) theory will be broken if $t \lesssim n\Lambda\Omega^{1-1/n}$. The approximated theory here is often called the linear perturbation and the basic requirement is that the t should not be larger than $n\Lambda\Omega^{1-1/n}$, e.g., $t \lesssim t_{\max} \approx sn\Lambda\Omega^{1-1/n}, s \ll 1$. Moreover, if one considers the next contribution to the solution, i.e., $x(t) = \Omega^{1/n}(1+\alpha t+\beta t^\sigma)$, where $\alpha = \Omega^{1/n-1}/n\Lambda$, then after some straightforward calculations, one obtains that $\sigma = 2$ and consequently,

$$x(t) \approx \Omega^{1/n}\left(1+\frac{1}{n\Lambda}\Omega^{1/n-1}t - \frac{n-3}{2n^2\Lambda^2}\Omega^{2/n-2}t^2\right). \tag{2.3}$$

One has that $t \ll n\Lambda\Omega^{1-1/n}$ from the first-order theory. If higher order corrections are taken into consideration, e.g., the second-order contribution appeared in (2.3), the condition for the perturbation theory is obtained as $|(n-3)\delta^2(t)/2| \ll |\delta(t)|$, or equivalently $|(n-3)\delta(t)/2| \ll 1$, or $t \ll 2n\Lambda\Omega^{1-1/n}/(n-3)$. It becomes $t \ll 2\Lambda\Omega^{1-1/n} \sim 2\Lambda\Omega$ if n is large, which is weaker than the criterion $t \ll n\Lambda\Omega^{1-1/n}$. indicating the effective perturbative region shrinks as the order of the expansion increases.

The perturbation element (or the small quantity in general) of the Eq. (2.1) is $\delta(t) = \alpha t = \Omega^{1/n-1}t/n\Lambda$, and consequently $x(t) \approx \Omega^{1/n}[1+\delta(t) - (n-3)\delta^2(t)/2]$. It is obvious that it could not be treated as small when the t is large, indicating that either the linear theory or the theory with higher order terms breaks down at large t. However, on the other hand, in the limit that the t approaches to infinity, another perturbative scheme for Eq. (2.1) emerges. In that situation, the term Ω on the right hand side of the equation (2.1) could be safely neglected, leading to $x_\infty(t) = (t/\Lambda)^{1/(n-1)}$, and it is called the asymptotic solution (large-t) of Eq. (2.1). Assuming that $x(t) \approx x_\infty(t)[1+\phi(t)]$ based on the asymptotic solution and the factor $|\phi(t)| \ll 1$, one could obtain,

$$x(t) \approx \left(\frac{t}{\Lambda}\right)^{\frac{1}{n-1}}\left[1+\frac{\Omega}{n-1}\left(\frac{\Lambda}{t}\right)^{\frac{n}{n-1}}\right], \quad \phi(t) = \frac{\Omega}{n-1}\left(\frac{\Lambda}{t}\right)^{\frac{n}{n-1}}, \tag{2.4}$$

with the condition that

$$t \gg t_{\rm asp} \equiv \Lambda \exp\left(-\frac{n-1}{n}\log\frac{n-1}{\Omega}\right). \tag{2.5}$$

Moreover, considering that $x(t) \approx x_\infty(t)[1+\phi(t)+\mu(t)]$ to even higher order with $\mu(t)$ the contribution smaller than $\phi(t)$, we have

$$x(t) \approx \left(\frac{t}{\Lambda}\right)^{\frac{1}{n-1}}\left[1+\frac{\Omega}{n-1}\left(\frac{\Lambda}{t}\right)^{\frac{n}{n-1}} - \frac{n\Omega^2}{2(n-1)^2}\left(\frac{\Lambda}{t}\right)^{\frac{2n}{n-1}}\right], \tag{2.6}$$

and thus $x(t) \approx x_\infty(t)[1+\phi(t) - n\phi^2(t)/2]$, i.e., $\mu(t) = -n\phi^2(t)/2$.

EXERCISE 1: Derive the analytic expressions for the $x(t)$ of Eq. (2.1) to order $\delta^3(t)$ and $\phi^3(t)$. Discuss their applicable conditions.

EXERCISE 2: Consider the equation by generalizing Eq. (2.1) to be $x^n(t) = \Omega + tx^m(t)/\Lambda$ with $m < n$, develop its approximated solutions.

The exact solution of the algebraic equation (2.1) could be numerically constructed via, e.g., the algorithm,

$$x^{(j+1)}(t) = \left(\Omega + \frac{tx^{(j)}(t)}{\Lambda}\right)^{1/n}, \tag{2.7}$$

starting from an initial value $x^{(0)}$ for iteration. In Fig. 2 we show an example of the solution of Eq. (2.1) where $n = 6$ and $\Lambda = \Omega = 1$ are adopted, and consequently one obtains $t_{\rm asp} \approx 0.26$. It is shown obviously from the figure that after combining the asymptotic solution with the approximation developed with small t, one could construct the total solution of Eq. (2.1) eventually (for example, the region bounded by the small-t approximation to second order (shown with the blue line) and the asymptotic solution to second order (shown with the cyan line)), and moreover one could always do this better by considering higher and higher orders both in the small-t and in the asymptotic regions. In addition, the convergence structure of the solution (either small-t or the asymptotic situation) is commutative in the sense that the deviation from the exact one is commutative. For the problem (2.1) if $n = 3$, the second-order term in the small-t expansion is zero, indicating that even higher order terms may play similar role on the solution and one consequently need to investigate these terms carefully. Furthermore, the asymptotic solution is inversely correlated with the t and in this sense we have developed effective perturbative calculations on the small quantity $1/t$ when t itself is large. The perturbative relation on the t and/or on the $1/t$ is called the duality of the problem. Finally the medium region, for instance the region bounded by the magenta (blue) and the red (cyan) lines, is often non-perturbative and the solution of the problem in this region generally depends on some numerical recipe such as (2.7), although the the area of the medium region eventually decreases as the order of perturbation increases. For example, the applicable region for the first-order theories by setting $x^{\rm 1st}_{{\rm small}-t}(t) \approx x^{\rm 1st}_{{\rm large}-t}(t)$ is found to be about $0 \lesssim t \lesssim 1.23$, while that for the second-order theories is about $0 \lesssim t \lesssim 1.36$, see the green circles in Fig. 2. Although the curves are closer to the exact one, the improvement on the applicable region is not large.

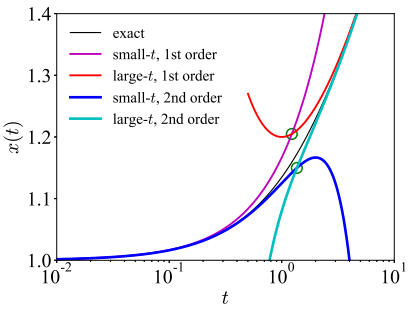

Fig. 2: Solution of Eq. (2.1), $n = 6, \Lambda = \Omega = 1$.

EXERCISE 3: Compute the area of Fig. 2 formed by the exact curve, the large-t and the small-t curves to first order and second order.

III. EXAMPLE: PERIOD OF A SIMPLE PENDULUM

As we all know that the period of a simple pendulum with a small swing angle (e.g., $\lesssim 5^\circ$) is approximately given by $T = 2\pi(l/g)^{1/2}$, where l is length of the pendulum, g is the gravity acceleration constant. When writing down this formula one assumes that the pendulum is ideal in the sense that there is no damping and external driven forces and also the maximum swing angle is small, i.e., $\chi_{\rm max} \lesssim 5^\circ$. For the simple pendulum, the physical quantities involved are the length (l), the mass of the ball (m), the gravity acceleration constant (g) and the maximum swing angle ($\chi_{\rm max}$). Here the maximum swing angle $\chi_{\rm max}$ is dimensionless and it should appear only in the "number" terms without any physical dimensions. Moreover, as the swing angle becomes larger and larger, the period of the system increases. For instance, the period is expected to be infinity in the case that the swing angle is π. Based on these analyses, the maximum angle $\chi_{\rm max}$ is expected to appear in the period of the pendulum in the following form,

$$T = 2\pi(l/g)^{1/2} \times \left(1 + a\chi^2_{\rm max} + \cdots\right), \tag{3.1}$$

and moreover the coefficient a is positive.

EXERCISE 4: Derive the zeroth-order term of (3.1), i.e., $2\pi(l/g)^{1/2}$, via solving the equation of motion of the system with small angle.

EXERCISE 5: If a quantity f has the dimension as $[{\rm M}]^\mu[{\rm L}]^\nu[{\rm T}]^\sigma$ with M, L and T the basic mass, length and time units, could μ, ν and σ be any real numbers? Explain it or give counterexamples.

We often denote the simple pendulum with the maximum swing angle $\chi_{\rm max} \gtrsim 5^\circ$ the non-harmonic system and the one with $\chi_{\rm max} \lesssim 5^\circ$ the harmonic system. The harmonic property of the system depends in fact on the force (or equivalently the potential), e.g., for the simple pendulum this force is given by the tangent component of the gravity $f = -mg\sin\chi$. In the situation that $\chi \lesssim \chi_{\rm max} \lesssim 5^\circ$, one naturally has $\sin\chi \approx \chi, f \approx -mg\chi$, indicating that the corresponding potential is proportional to χ^2, i.e., it is harmonic. When writing down the formula (3.1), we actually use the fact — the pendulum has the left-right symmetry, and although this symmetry is very elementary it stills gives important information, namely there would be no terms proportional to the odd orders of $\chi_{\rm max}$.

In order to obtain the period of a harmonic system, one naturally needs firstly to discuss the stability and the equilibrium and non-equilibrium issues. In the stable equilibrium problem, the particle could come back to the minimum if it is slightly far from the ground state, i.e., the periodic motion is formed. The basic method for calculating the period is Taylor's expansion theory, which is also the most important cornerstone of all the perturbative calculations. The motion with slight deviation from the equilibrium state indicates that the quantity $\delta x = x - X$ is small such that the potential U could be expanded around X according to (4.1). Selecting the equilibrium position at $X = 0$ (since the zero point of the potential has no effects on the dynamical processes), and moreover adopting that $U(X) = 0$, one obtains $U(x) = 2^{-1}U''(0)x^2, \delta x \to x$, which is fundamentally a parabolic equation (originating from the second-order approximation

of the potential). On the other hand, the conservation of the mechanical energy leads to $2^{-1}mv^2 = E - 2^{-1}U''(0)x^2$ from which one obtains

$$T = 2\pi\sqrt{m/U''(0)}. \tag{3.2}$$

For instance, for the harmonic oscillator the potential is given by $U = kx^2/2, U''(0) = k$, then consequently the period is $T = 2\pi(m/k)^{1/2}$, where m is the mass of the oscillator. It is necessary to point that the formula (3.2) is exact for the harmonic system.

For the general motion of the simple pendulum with any $\chi_{\max}$, the period is given by,[1]

$$T \approx T_0 \sum_{j=0}^{\infty} \Delta_j \chi_{\max}^{2j}, \tag{3.3}$$

where the zeroth order period is $T_0 = T(\chi_{\max} = 0) = 2\pi\sqrt{l/g}$, and $\Delta_0 = 1$, $\Delta_1 = 1/16$, $\Delta_2 = 11/3072$ and $\Delta_3 = 173/737280$ are for the first three corrections. In Fig. 3, we show the reduced period of the system $T(l/g)^{-1/2}/2\pi$ as a function of $\chi_{\max}$ at different orders of $\chi_{\max}$. If $\chi_{\max} = \pi$ the integration $\int_0^1 \mathrm{d}\vartheta[1-\sin^2(\vartheta)]^{-1/2}$ appearing in the calculation is divergent, indicating the perturbative scheme is now broken down. From the physical viewpoint, the case $\chi_{\max} = \pi$ corresponds to the unstable equilibrium state of the system and consequently it has no period (or the period is infinity). Furthermore, the coefficients $\Delta_1, \Delta_2, \Delta_3$, etc., guarantee that each term in the expansion of the period is a perturbation compared with its previous term although $\chi_{\max}$ could essentially be larger than 1.

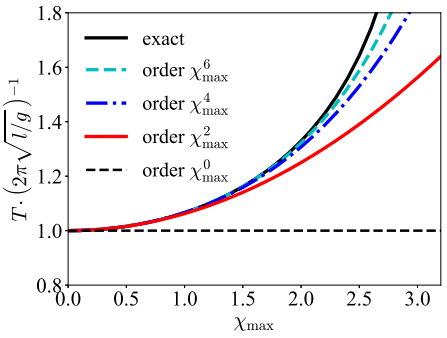

Fig. 3: Reduced period of the pendulum as a function of the maximum angle $\chi_{\max}$.

EXERCISE 6: Assume the equation for χ is $\ddot{\chi} + \phi\chi - \phi\chi^3/6 = 0$, where $\phi > 0$ is a constant. If χ is a small quantity $\chi \ll 1$, the last term on the left hand side could be treated as a perturbation and in this situation analytically solve the above equation for $\chi(t)$ under initial conditions $\chi(0) = 0$ and $\dot{\chi}(0) = C$. Then solve it when the last term is included numerically.

EXERCISE 7: One of the root of the simple equation $ax^2 + bx + c = 0$ with $abc \neq 0$ and $b > 0$ is given by

$$x^* = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{b}{2a}\left(\sqrt{1 - \frac{4ac}{b^2}} - 1\right). \tag{3.4}$$

Assume that a is small in the sense $k = 4ac/b^2 \ll 1$, try to obtain the approximation for x^* from (3.4) by expanding the square root to order k^2 using the formula $\sqrt{1+w} \approx 1 + w/2 - w^2/8$ for small w. The same result could also be obtained via firstly solving the linear equation $bx + c = 0$ and then adding some perturbation p to the solution $-c/b$ as $x = -(c/b)(1+p)$. Determine the expression for p to first order.

Finally we discuss the period of a harmonic system. In fact, the period of a harmonic system could be obtained by analyzing the structure of its total energy. Specifically, If the total energy of a system could be written as $E(\beta, \dot{\beta}) = f\dot{\beta}^2 + g\beta^2 + C$, where β is the generalized coordinate. Harmonic oscillation (motion) requires that $E - C = f\dot{\beta}^2 + g\beta^2 > 0$.[2] The frequency, period, and the amplitude of β of the system are

$$\omega = \left(\frac{g}{f}\right)^{1/2}, \quad T = 2\pi\left(\frac{f}{g}\right)^{1/2}, \quad \beta_{\max} = \left(\frac{E-C}{g}\right)^{1/2}, \tag{3.5}$$

respectively, here the energy E is a conservative quantity of the system considered. Usually one needs to analyze the force the system underwent and then establish the equation of motion, and finally by solving the equation of motion to obtain the period of the system, see EXERCISE 4. The above approach provides a convenient tool to obtain the period of the system. Often calculating energy is some easier than solving the equation of motion. However it is also essential to point that there is no terms proportional to $\beta\dot{\beta}$ (i.e., the cross terms) in the expression of the system energy, i.e., the energy contains only the quadratic terms in the generalized coordinate, the generalized velocity and some constants. These coordinates are called the normal coordinates.

EXERCISE 8: Apply formula (3.5) to the small-angle pendulum system, and derive the expression for the period of the double-pendulum system with equal mass/length moving in a plane.

IV. NONLINEAR EFFECTS: NOVELTY AND COMPLEXITY

In this section, let us show how to do general calculations for the harmonic system including the nonlinear effects. Consider the particle (as the ball in simple pendulum) moves under the potential shown in Fig. 4 where X is the minimum of the potential. For motion around the minimum X, the potential acting on the particle could be approximated by expanding the potential $U(x)$ as,

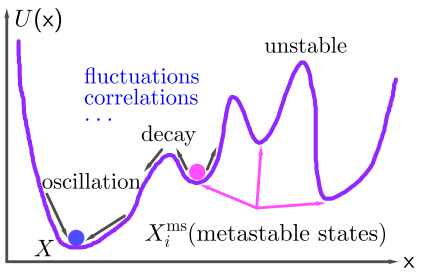

Fig. 4: Potential $U(x)$ near the minimum X where $U'(X) = 0$ and $\omega = U''(X) > 0$.

$$U(x) \approx U(X) + \dot{U}|_{x=X}\delta x + 2^{-1}\ddot{U}|_{x=X}\delta x^2, \tag{4.1}$$

where $\delta x^n = (x - X)^n$, $\dot{U} = \mathrm{d}U/\mathrm{d}x$, and $\ddot{U} = \mathrm{d}^2U/\mathrm{d}x^2$, and since the first-order derivative of the potential at the equilibrium X is zero, one obtains

$$U_{\rm harm}(\delta x) = 2^{-1}\omega^2\delta x^2 + \text{const.}, \quad \omega = [U''(X)]^{1/2}, \tag{4.2}$$

where the constant is the zero point of the potential (which actually has no fundamental effects on the dynamics processes). The above one is called the harmonic potential, and the solution of which could be obtained exactly. Now, if one tries to study the behavior of the particle far from the equilibrium position X, the natural treatment is investigating the effects from the high order terms (e.g., the term δx^3) perturbatively based on the harmonic solution. This is the frequently-used method in physical problems: Firstly obtaining the solution via the simple approximation (here it is given by $U_{\rm harm}(\delta x)$, the terms like this are often called the non-interacting terms), and then perturbatively computing the high order effects based on the simple solution. Furthermore, the oscillation around the meta-stable stats $X_i^{\rm ms}$ is also important and the transition from the meta-stable states to the global ground state is one of the most exciting problems in modern field calculations.[3]

[1] Some details of the derivation of (3.3): The equation of motion of the pendulum is $2^{-1}mv^2(t) + U(x,t) = E, v(t) = \mathrm{d}x/\mathrm{d}t$. Solving it gives $\mathrm{d}x/\mathrm{d}t$, and consequently the time expressed in terms of the integration, i.e.,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{\frac{2[E - U(x)]}{m}}, \quad t = \sqrt{\frac{m}{2}}\int \frac{\mathrm{d}x}{\sqrt{E - U(x)}}.$$

The period of the system is consequently given as

$$T = 4\sqrt{\frac{m}{2}}\int_0^{\chi_{\max}} \frac{\mathrm{d}x}{\sqrt{E - U(x)}} = 4\sqrt{\frac{l}{g}}\int_0^{\pi/2} \mathrm{d}\vartheta\left(1 - \sin^2\frac{\chi_{\max}}{2}\sin^2\vartheta\right)^{-1/2}.$$

For the zeroth-order approximation, $\chi_{\max} \approx 0$ (small angle), then one obtains the familiar result $T = 4\sqrt{l/g}\int_0^{\pi/2}\mathrm{d}\vartheta = 2\pi\sqrt{l/g}$. Similar results could be obtained in the first-order theory and so on.

[2] This form of the energy is often obtained from the linear response theory. The g in non-harmonic systems could be negative and the form of the potential energy even be non-quadratic, e.g., the gravity potential $U_{\rm G}(r) = -GM/r$.

[3] In modern physics, the tunneling between the true vacuum state and the false vacuum is an important issue, see, e.g., S. Coleman, *Fate of the False Vacuum:*

Consider the extra force $f^\delta(x)$ based on Hooke's force in the harmonic system, $\delta x \to x$. In this situation the energy conservation equation becomes $2^{-1}m\dot{x}^2 + 2^{-1}kx^2 + U^\delta(x) = 2^{-1}kX^2 + U^\delta(X)$, where X the maximum distance the oscillator could reach, and $U^\delta(x)$ is the potential due to the extra force. If the extra potential is homogeneous with order α, i.e., $U^\delta(\lambda x) = \lambda^\alpha U^\delta(x)$, the period of the system is given by

$$T = 4\sqrt{\frac{m}{k}}\int_0^{\pi/2} \mathrm{d}\phi \left[1 + \frac{2U^\delta(X)[1-\sin^\alpha\phi]}{kX^2\cos^2\phi}\right]^{-1/2}. \tag{4.3}$$

In certain situations one can do perturbative calculations for the relevant quantities by expanding around the $\xi = 2U^\delta(X)[1-\sin^\alpha\phi]/kX^2\cos^2\phi$.

As an elementary example, taking the extra force as $f^\delta(x) = -ax^3 (a > 0), \xi = (aX^2/2k)(1+\sin^2\phi)$, the effective potential is then given by $U_{\rm eff}(x) = U_{\rm tot}(x) = 2^{-1}kx^2 + 4^{-1}ax^4$ (the effective potential like this has actually little use since the high order term here still contains the dynamical variable "x"), characterizing the cubic response to the perturbation. One could obtain to order a^2 that,

$$T \approx 2\pi\sqrt{\frac{m}{k}} \times \left(1 - \frac{3aX^2}{8k} + \frac{57a^2X^4}{256k^2}\right), \tag{4.4}$$

and the perturbative condition is $aX^2/k \ll 1$, or equivalently $U^\delta(X) \ll U(X)/2$. In this formula, if the following replacements, i.e.,

$$m/k \leftrightarrow l/g, \;\; X = \chi_{\rm max} l, \;\; a = -mg/6l^3, \tag{4.5}$$

are adopted, one immediately obtains the first-order correction coefficient 1/16 in the period (3.3) of the simple pendulum. However the even higher order corrections could not be obtained simply through (4.4), since the period of the simple pendulum contains the higher order corrections from $\sin\chi$ (in fact the extra force $f^\delta(x) = ax^3$ is similar as the cubic term χ^3 of the pendulum). The correction directly from the term ax^3 corresponds to the conventional perturbation theories in field problems, and the one characterized by the correction factor $\zeta(\chi_{\rm max}) = 1 + 16^{-1}\chi_{\rm max}^2 + (11/3072)\chi_{\rm max}^4 + \cdots$ corresponds to the improved perturbations, which are sometimes called the resummations.

The fourth-order correction $(11/3072)\chi_{\rm max}^4$ in the period of the simple pendulum could be decomposed into two terms: the $\chi_{\rm max}^2$ term and the $\chi_{\rm max}^4$ term from the interacting energy $E(\chi_{\rm max}) = -mgl\cos\chi_{\rm max} \approx -mgl(1 - 2^{-1}\chi_{\rm max}^2 + 24^{-1}\chi_{\rm max}^4 + \cdots)$, or equivalently the terms proportional to χ and to χ^3 in the force $F(\chi) = -mg\sin\chi \approx -mg(\chi - 6^{-1}\chi^3)$. One could obtain the corresponding nonlinear effects simply by considering the χ^3 term based on the harmonic approximation, but the coefficient is 19/3072 (via the formula (4.4)) instead of 11/3072. Resummation indicates that besides the "direct" term — χ^3, the higher order term originated from χ (e.g., the first term in $\sin\chi \approx \chi - \chi^3/6$) also contributes to the coefficient 11/3072. This latter one is denoted as the "indirect" contribution. In other words, there exists the mode-coupling between the low modes (here characterized by χ) and the high modes (characterized by χ^3), i.e., $\chi^3 \leftarrow (\chi^3)^1$(direct term)$+(\chi^1)^3$(indirect high order terms). As the index"n" appearing in χ^n becomes large, the mode-coupling pattern will also become more fruitful.

At this point it is necessary to introduce the very basic concept of effective theories. In the case of the aforementioned example on the Hooke system with an extra force ax^3, one could derive an effective Hooke's constant through the period of the system. In particular, according to the period formula (3.2), one has $T = 2\pi(m/k_{\rm eff})^{1/2} = 2\pi m^{1/2}k_{\rm eff}^{-1/2}$ where the effective spring constant is $k_{\rm eff} \approx k(1 + s_1\sigma + s_2\sigma^2) + \mathcal{O}(\sigma^3)$ with $\sigma = aX^2/k \ll 1$. In order to reproduce the first two terms as shown in the formula (4.4), i.e., $-3\sigma/8$ and $57\sigma^2/256$, one obtains the two coefficients as $s_1 = 3/4, s_2 = -3/128$. The effective Hooke's constant is thus

$$k_{\rm eff} \approx k \times \left(1 + \frac{3}{4}\sigma - \frac{3}{128}\sigma^2\right), \tag{4.6}$$

and in other applications one could use the effective potential $U_{\rm eff}(x) = 2^{-1}k_{\rm eff}x^2$ to do the calculation. Here the high order effects characterized by the coefficient a appears in the effective potential without the "dynamical" variable "x", and the small quantity $\sigma = aX^2/k$ is fundamental for the effective theory.

EXERCISE 9: Assume the effective spring constant is $k_{\rm eff} \approx k(1 + s_1\sigma + s_2\sigma^2 + s_3\sigma^3 + s_4\sigma^4)$, work out the value of s_3 and s_4.

However, there exist other approaches to construct the effective parameters, e.g., the Hooke's constant in the presence of the nonlinear force could also be obtained as $\overline{k}_{\rm eff} = k + aX^2/2$ by considering the maximum distance, indicating other mechanisms need to be taken into account in the construction of an effective theory. We have no attempt to introduce/discuss these advanced issues in the present lecture. The effective theories with the high order degrees of freedom integrated out are often called the "low-energy effective theories". The maximum distance X could further be expanded around X_0 which is the maximum distance without the extra force, see (6.5), and the construction of the effective spring constant could become even more complicated, see expressions (6.6) and (6.12). Section VI gives further calculations on these issues.

V. SPONTANEOUS SYMMETRY BREAKING: BASIS

It is straightforward to solving the time dependence of $\chi(t)$ for the simple pendulum problem, and however one could also investigate the relation between $\chi(t)$ and $\dot{\chi}(t)$, the two-dimensional plane determined by χ and $\dot{\chi}$ the phase space (plane). The curve in phase space is called the phase curve (phase diagram). The relevant point is that one can study the dynamical features of the system directly from its phase curve, instead of solving the differential equation. The energy conservation for the pendulum could be rewritten as $2^{-1}\dot{\chi}^2 + 1 - \cos\chi = \widetilde{E}$ where $\widetilde{E} = E/mgl$, and $\omega = (g/l)^{1/2} = 1\,\mathrm{s}^{-1}$. It is a dimensionless equation involving the χ and $\dot{\chi}$, and the relation $\chi \sim \dot{\chi}$ predicted is shown in Fig. 5, where the curve for the critical value $\widetilde{E} = 2$ is also shown with magenta circles.

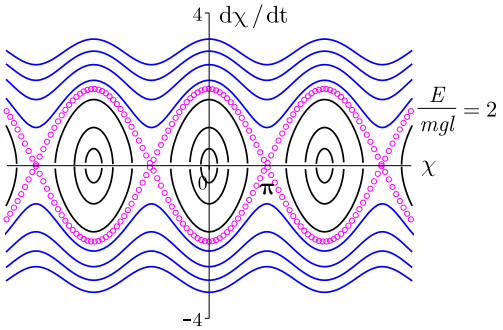

Fig. 5: Phase diagram of $\dot{\chi} \sim \chi$ where the points $\pm\pi$ denote the instability. The phase orbit is closed if $\widetilde{E} < 2$ and the angle χ can not take the values $\pm\pi$. The area bounded by the closed orbit is characterized by the energy of the system, i.e., as the energy of the system increases the area increases, indicating a large period.

It is obvious from Fig. 5 that if the reduced energy $\widetilde{E} < 2$, the phase curve is closed, indicating that the motion of the simple pendulum is periodic. Moreover, as the energy increases, the area bounded by the curve also increases. On the other hand, the orbit is close to the circle when the energy is small and as the energy eventually increases the orbit approaches to ecliptic and finally when the $\widetilde{E}$ equals to 2, the orbit undergoes the fundamental distortion to the one shown by the magenta circle. In the critical situation that $\widetilde{E} = 2$, the two orbits are cross with each other and separate the total phase diagram into different regions. When $\widetilde{E} > 2$, the orbits are well separated either in the upper or in the lower parts and moreover these orbits are not closed, indicating that the swing angle decreases or increases monotonically. More specifically, the pendulum now rotates around some direction which depends on the initial velocity. Furthermore, as the energy increases even more the phase orbit becomes flatter and more distant from the horizontal axes. The period of the χ is 2π, and the two values for χ, i.e., $\pm\pi$, are actually the same position. Finally, if one glues the two points $\pm\pi$, i.e., transforming the phase plane into a cylindrical one, all the phase orbits will

Semiclassical Theory, Phys. Rev. D **15**, 2929 (1977); C. Callan and S. Coleman, *Fate of the False Vacuum. II: First Quantum Corrections*, Phys. Rev. D **16**, 1762 (1977). Discussions on topics of these papers are beyond the present lecture.

emerge on the surface of this cylindrical plane, and the periodic motion will be limited within the boundary lines and the rotations will go away around the cylindrical surface. For the critical situation, $\dot{\chi} = \pm 2\cos\frac{\chi}{2}$, one could further obtain $\dot{\chi} = 2/\cosh t$, which is called the soliton solution of the system. Physically the soliton solution could be understood as the time-reverse state of the original one.

The nonlinear effects are generally important especially in some certain situations, and some properties of the system could naturally demonstrated via the nonlinear terms, which are often absent in the harmonic system. Here we would like to introduce the very important concept of the symmetry breaking via the inverted pendulum. In the inverted pendulum, the ball with mass m is fixed by the spring at the equilibrium angle $\beta = 0$. The ball moves under the influence of the gravity and the elastic forces. Moreover, the torque provided by the spring is given by $M = -\kappa\beta$ with κ the Hooke's constant for the spring. Consequently the elastic potential is $U_\kappa(\beta) = \kappa\beta^2/2$, while the gravity potential is given by $U_G(\beta) = mgl(\cos\beta - 1)$ (with the zero point selected at the top of the system). Then the total potential energy is the sum of these two terms, i.e., $U(\beta) = U_\kappa(\beta) + U_G(\beta) = 2^{-1}\kappa\beta^2 + mgl(\cos\beta - 1)$.

The equilibrium state β_0 is straightforwardly obtained by the first-order derivative of the potential, i.e., $\kappa\beta - mgl\sin\beta = 0$. The critical value for the length is given as $l_c = \kappa/mg$ by calculating the second-order derivative of the potential, i.e., $U''(\beta) = \kappa - mgl\cos\beta = mg(l_c - l\cos\beta)$. The phenomenon that the length l varies continuously from the one smaller than l_c (then $\beta_0 = 0$ is stable) to the one larger than l_c (then $\beta_0 = 0$ is unstable) and in the mean while the situation with only one oscillation center (at $\beta_0 = 0$) evolves into the one with two oscillation centers (at $\beta_0 \neq 0$ with β_0 given by the non-zero solution of the equation $\kappa\beta - mgl\sin\beta = 0$), strongly indicating that the the structure of the phase diagram undergoes some fundamental (topological) changes, is generally called the bifurcation. In the case that $l > l_c$ although the overall phase diagram is symmetric, the particle (ball) could only oscillate near any one of these two equilibrium positions. From the viewpoint of classical mechanics the ball could not penetrate from one equilibrium state into the other, indicating some breaking of the original symmetry occurs. On the other hand, there still exists some possibilities that the particle could penetrate from one equilibrium state into the other, and this phenomenon is called the tunning in quantum mechanics. The above phenomenon, i.e., the symmetry pattern of the system undergoes some sudden change due to the external parameters (like l_c), is generally called the spontaneously symmetry breaking (SSB). After introducing the parameter $\theta = \kappa/mgl$, the total potential of the inverted pendulum could be rewritten into the dimensionless form, $\widetilde{U}(\beta) \equiv U(\beta)/mgl = 2^{-1}\theta\beta^2 + \cos\beta - 1$, with its curve shown in Fig. 6. The SSB occurs if $\theta < 1$, i.e., the symmetry of the ground state (with the smallest potential energy) is lower than that of the potential itself.

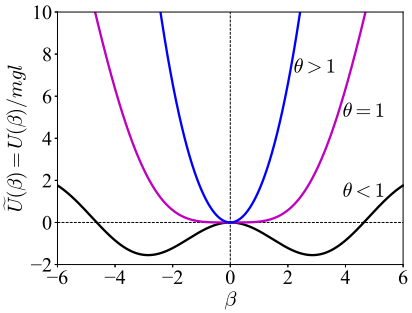

Fig. 6: The reduced potential energy $\widetilde{U}(\beta)$ as a function of β with different $\theta = \kappa/mgl$, and $\theta_c = 1$ is its critical value. If $\theta < \theta_c$, the SSB occurs and two possibilities are: a) the length of the system is very large; b) the Hooke's constant κ is very small. In both cases, the SSB indicates that the oscillations near the two equilibrium states are independent.

The ground state of the inverted pendulum is easily determined via $\partial U(\beta)/\partial\beta|_{\beta=B} = 0$, and if β is small one has $B = [6(1-\theta)]^{1/2}$ where without losing generality the ground state is selected at the right equilibrium position. If one introduces the fluctuation χ around the ground state by $\beta = B + \chi$, one could obtain the potential for the fluctuation as

$$\widetilde{U}(\chi) \approx -\sqrt{\frac{27}{50}}(1-\theta)^{5/2}\chi + \frac{1}{4}(1-\theta)(1+3\theta)\chi^2 + \theta[(1-\theta)/6]^{1/2}\chi^3 + \widetilde{U}_0, \quad (5.1)$$

where $\widetilde{U}_0 = 3\theta^3/16 - 12\theta^2/5 + 39\theta/10 - 9/5$. Due to the cubic term χ^3, the potential $\widetilde{U}(\chi)$ already has no the symmetry "$\chi \leftrightarrow -\chi$", demonstrating that the symmetry is broken. Moreover, the term $4^{-1}(1-\theta)(1+3\theta)\chi^2$ could be understood as the mass term of the χ.

EXERCISE 10: Work out expressions for the 4th-order and 5th-order self-interactions for the χ field by expanding $\widetilde{U}(\chi)$ to χ^5.

EXERCISE 11: Design numerical algorithm to solve $\theta\beta = \sin\beta$ for $0 < \theta < 1$, and develop first-order correction to the solution $\sqrt{6(1-\theta)}$.

Assuming β is small, one then has $\widetilde{U}(\beta) \approx 2^{-1}(\theta - 1)\beta^2 + 24^{-1}\beta^4$. The β plays some similar roles as the Higgs particle in modern theoretical physics. For example, via the SSB of the self-interaction of the Higgs' field $U(\phi) = -2^{-1}\mu^2\phi^2 + 4^{-1}\lambda\phi^4$ ($\mu \in \mathrm{R}, \lambda > 0, \hbar = 1$), one could obtains the mass of the Higgs' particle as $m_H = (2\lambda\langle\phi\rangle^2)^{1/2}$, with the average squares given by $\langle\phi\rangle^2 = \mu^2/\lambda$. The SSB in the inverted pendulum is characterized by the term $\theta(1-\theta)^{1/2}6^{-1/2}\chi^3$. The simplest unsymmetric potential is shown in Fig. 7, where $|X_1| < |X_2|$, e.g., $U(x) = n_1x^3 + n_2x^2$ with $n_1 = U_0(\nu-1)/\nu^2X_1^3, n_2 = U_0(\nu^2 - \nu + 1)/\nu^2X_1^2$, and $\nu = -X_2/X_1$. Based on this potential, several interesting dynamical quantities could be investigated, e.g., one can treat the cubic interaction as a perturbation and do relevant calculations, alternatively another perturbative scheme based on $\nu = 1 + f$ with f small could be developed.

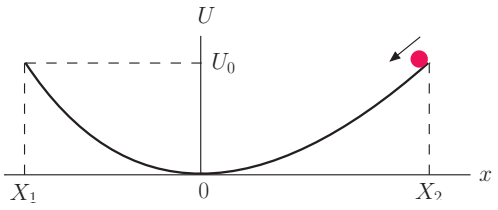

Fig. 7: Unsymmetric potential with $|X_1| < |X_2|$.

EXERCISE 12: Work out the period of the motion in potential of Fig. 7 upto quadratic order of f. Demonstrate the range of ν and then show f could really be treated as perturbatively.

VI. PERTURBATION THEORIES: IMPLICIT FORM

In some problems it is very difficult to solve the quantity involved explicitly and then do the perturbative calculations, e.g., the solution of the algebraic equation (2.1). In these situations some general guidance is as follows: Perturbatively expanding the two sides of the equation (often the definition of the physical quantity involved) at the same time (this step avoids the solution of the equation), and then match the terms on both sides order by order. We call the calculation like that the "integration form" (or the implicit form) of the perturbation. On the other hand, the approach first obtaining the explicit expression of the relevant quantity and then expanding it around some fixed point the "differential form" of the perturbation. Generally, one can think that the "differential form" of the perturbation method is as a special case of the "integration forms" of the perturbation method, with the latter providing certain convenience from treating complicated problems.

We now use the harmonic oscillator with the nonlinear effects to calculate the critical time t_c and the maximum compressed distance L_{max} between to colliding balls. This is a typical example of the "integration form" of the perturbation theories. The maximum compressed distance L_{max} and the corresponding critical time t_c are two fundamental quantities characterizing the compression process, and moreover these two quantities, i.e., L_{max} and t_c, are closely related via the interaction between the two balls, thus it is very interesting to investigate how the nonlinear force (besides the conventional Hooke's linear force) affect these quantities. The simplest

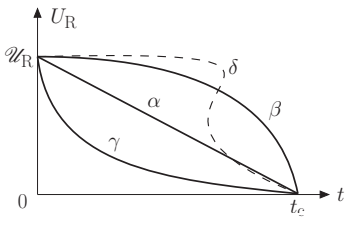

Fig. 8: Time dependence of relative velocity of the head-head collision.

kinetic description is as follows: Two balls with initial relative velocity $\mathscr{U}_{\rm R} = u_1 - u_2$ (here $u_1 > u_2$) collide with each other head to head reach the maximum compressed distance $L_{\rm max}$ at critical time $t_{\rm c}$, the corresponding compressed velocity is denoted by $V_{\rm c}$. See Fig. 8 for sketch of the time dependence of relative velocity $U_{\rm R}$.

The simplest form for the relative velocity is thus $U_{\rm R}(t) = \mathscr{U}_{\rm R} + At$, where A is an effective acceleration (which is negative), the compressed distance is straightforwardly obtained as $L(t) = \int_0^t U_{\rm R}(t){\rm d}t = \mathscr{U}_{\rm R}t + 2^{-1}At^2$. Consequently, the critical time is given by $t_{\rm c} = -\mathscr{U}_{\rm R}/A$, and the maximum compressed distance is $L_{\rm max} = L(t_{\rm c}) = 2^{-1}\mathscr{U}_{\rm R}t_{\rm c} = -\mathscr{U}_{\rm R}^2/2A$. These calculations are effectively corresponding to the α case shown in Fig. 8. In this situation the relation between the $L_{\rm max}$ and the critical time $t_{\rm c}$ is very simple, i.e., $L_{\rm max}/t_{\rm c} = 2^{-1}\mathscr{U}_{\rm R}$. The assumption on the constant effective acceleration A during the compression process is unrealistic, and in the general cases the relative velocity has the basic form $U_{\rm R}(t) = \mathscr{U}_{\rm R} + \mathscr{Q}(t)$, where $\mathscr{Q}(t)$ is a function of the time t with initial value $\mathscr{Q}(0) = 0$ and $\mathscr{Q}(t) < 0$ if $t > 0$. The equation determining the critical time is $U_{\rm R}(t_{\rm c}) = 0$, i.e., $\mathscr{U}_{\rm R} + \mathscr{Q}(t_{\rm c}) = 0$. Correspondingly,

$$L_{\rm max} = -t_{\rm c}\mathscr{Q}(t_{\rm c}) + \int_0^{t_{\rm c}} \mathscr{Q}(t){\rm d}t, \tag{6.1}$$

which only depends on the $t_{\rm c}$ and the relative velocity $\mathscr{Q}$.

For example, if $\mathscr{Q}(t) = -\beta t^n$, where $\beta > 0$ and $n > 1$, see the β case shown in Fig. 8, we now have the critical time and the maximum compressed distance,

$$t_{\rm c} = \left(\frac{\mathscr{U}_{\rm R}}{\beta}\right)^{1/n}, \quad L_{\rm max} = \frac{n\beta}{n+1}\left(\frac{\mathscr{U}_{\rm R}}{\beta}\right)^{1+1/n} = \frac{\mathscr{U}_{\rm R}t_{\rm c}}{1+1/n}. \tag{6.2}$$

Similarly, if the function $\mathscr{Q}(t)$ takes the form $\mathscr{Q}(t) = -\gamma\sin(t/T)$, where $T \geq 2t_{\rm c}/\pi, \gamma > \mathscr{U}_{\rm R}$ with T a constant, see the γ case shown in Fig. 8, then the critical time is given by $t_{\rm c} = T\arcsin(\mathscr{U}_{\rm R}/\gamma)$, and

$$L_{\rm max} = \mathscr{U}_{\rm R}t_{\rm c} + \gamma T\left[\sqrt{1-\left(\frac{\mathscr{U}_{\rm R}}{\gamma}\right)^2} - 1\right] \approx \frac{1}{2}\mathscr{U}_{\rm R}t_{\rm c}\left(1 - \frac{1}{12}\left(\frac{t_{\rm c}}{T}\right)^2\right). \tag{6.3}$$

If the velocity decreases uniformly/linearly from an initial value $\mathscr{U}_{\rm R}$, then the relation between the maximum compressed distance and the critical time is simply $L_{\rm max} = 2^{-1}\mathscr{U}_{\rm R}t_{\rm c}$. In the first example as n increases the in-front factor $1/(1+1/n)$ eventually becomes larger than 1/2 and approaches to the limit value of 1 as $n \to \infty$. In the second example, the in-front factor formally expressed as $2^{-1}[1 - 12^{-1}(t_{\rm c}/T)^2]$ is smaller than 1/2, which is different from that in the first example.

We assume that the interaction between the two balls when colliding with each other includes not only the Hooke's force but also the high order terms with the simplest case being that $f(x) = -kx - ax^3, a > 0$, see discussions of section IV. Consequently the energy conservation gives $2^{-1}\mu\mathscr{U}_{\rm R}^2 = 2^{-1}kX^2 + 4^{-1}aX^4$, where $X = L_{\rm max}^a, \mu = m_1m_2/(m_1+m_2)$ is the reduced mass of the two balls. One obtains

$$X = (k/a)\cdot\left(\sqrt{1 + 2a\mu\mathscr{U}_{\rm R}^2/k^2} - 1\right). \tag{6.4}$$

The parameter a is in general not a small quantity and the perturbatvie condition is $\sigma = 2a\mu\mathscr{U}_{\rm R}^2/k^2 \ll 1$, thus the above expression is the general solution. On the other hand if $2a\mu\mathscr{U}_{\rm R}^2/k^2 \ll 1$, then one could naturally make perturbative calculation for (6.4), leading to

$$L_{\rm max}^a \approx \left(\frac{\mu\mathscr{U}_{\rm R}^2}{k}\right)^{1/2}\left[1 - \frac{a\mu\mathscr{U}_{\rm R}^2}{4k^2} + \frac{7a^2\mu^2\mathscr{U}_{\rm R}^4}{32k^4}\right], \tag{6.5}$$

where $L_{\rm max} \equiv L_{\rm max}^0 \equiv X_0 = (\mu\mathscr{U}_{\rm R}^2/k)^{1/2}$ is the maximum distance in the Hooke case. Defining the effective Hooke's "constant" via $2^{-1}k_{\rm eff}X^2 = 2^{-1}kX^2 + 4^{-1}aX^4$ gives (see formula (4.6) for comparison),

$$k_{\rm eff} \approx k + \frac{a\mu\mathscr{U}_{\rm R}^2}{2k}\left[1 - \frac{a\mu\mathscr{U}_{\rm R}^2}{2k^2} + \frac{a^2\mu^2\mathscr{U}_{\rm R}^4}{2k^4}\right] = k\left(1 + \sigma - \sigma^2 + 2\sigma^3\right). \tag{6.6}$$

One could obtain the maximum compressed distance analytically in the above simple situation and however the analytical solution in the general situation is often very hard to obtain, i.e., solving the energy conservation equation is very difficult. Moreover, it is also very difficult to solve the differential equation to obtain the critical time $t_{\rm c}$ analytically even in this simplest case since the corresponding equation is given by $\mu\ddot{x} + kx + ax^3 = 0$. Now we are only interested in the case that the nonlinear interactions are perturbations, and assuming that the interaction takes the form as $f(x) = -kx - f^\delta(x)$, where x is the compressed distance and $f^\delta(x)$ the nonlinear force with $f^\delta > 0$. Moreover, assume that the potential corresponding to f is denoted by $U^\delta(X)$, where X is the maximum compressed distance. Consequently the energy conservation gives the equation $2^{-1}\mu\mathscr{U}_{\rm R}^2 = 2^{-1}kX^2 + U^\delta(X)$, from which one could determine the perturbative expression for the X. See section IV.

For example, if the extra force scales as $f^\delta \sim x^3$ (section IV), the energy conservation equation could be solved analytically. On the other hand, the analytical solution is impossible for general force f^δ. The energy conservation equation could be rewritten in the form $2^{-1}\mu\dot{x}^2 + U_{\rm tot}(x) = E$, where $E = \mu\mathscr{U}_{\rm R}^2/2$ is the total energy and $U_{\rm tot}(x) = kx^2/2 + U^\delta(x)$ is the potential energy during the compression process, solving it gives

$$t_{\rm c} = \frac{1}{\mathscr{U}_{\rm R}}\int_0^X {\rm d}\rho\left(1 - \frac{k\rho^2}{\mu\mathscr{U}_{\rm R}^2}\right)^{-1/2}\left[1 - \frac{2U^\delta(\rho)}{\mu\mathscr{U}_{\rm R}^2 - k\rho^2}\right]^{-1/2}. \tag{6.7}$$

If the nonlinear force is absent, i.e., $U^\delta = 0$, the above expression gives

$$t_{\rm c}^0 = \frac{1}{\mathscr{U}_{\rm R}}\int_0^X {\rm d}\rho\left(1 - \frac{k\rho^2}{\mu\mathscr{U}_{\rm R}^2}\right)^{-1/2} = \frac{\pi}{2}\sqrt{\frac{\mu}{k}}, \tag{6.8}$$

which is the solution for the free model.

For the general case involving the nonlinear force, one needs to use the "integration form" of the perturbative method. Assume that the characteristic perturbative quantity of the problem is d, and $X = X_0(1 + \Phi_1 d + \Phi_2 d^2 + \cdots)$. By putting it into the energy conservation equation and then comparing both sides of the equation of the terms d^i, one could gain the expressions for Φ_i's. The similar method can be applied to compute the critical time $t_{\rm c}$. For example, for the extra force scales as $f(x) = -dx^{2n+1}$ with $n > 1$ an integer, the energy conservation equation gives $2^{-1}\mu\mathscr{U}_{\rm R}^2 = 2^{-1}kX^2 + dX^{2n+2}/(n+2)$, where $X = L_{\rm max}^d$ is the maximum distance depending on the quantity d. Since there exists a characteristic factor d in front of the term X^{2n+2}, one approximates X^{2n+2} only to order d. Put them into the energy conservation equation and make the coefficients in front of d and d^2 zero gives the needed quantities Φ_1 and Φ_2, the final result to order d^2 for the maximum compressed distance is

$$X \approx X_0\left[1 - \left(\frac{1}{2n+2}\frac{\mu^n\mathscr{U}_{\rm R}^{2n}}{k^{n+1}}\right)d + \left(\frac{4n+3}{8n^2+16n+8}\frac{\mu^{2n}\mathscr{U}_{\rm R}^{4n}}{k^{2n+2}}\right)d^2\right]. \tag{6.9}$$

EXERCISE 13: Assume the force for the collision process is given by $F = -f_0(e^{\beta t/t_{\rm c}} - 1)$ for $0 \leq t \leq t_{\rm c}$, where $\beta, f_0 > 0$. Derive the expression for the critical time $t_{\rm c}$. Show that if $\beta \ll 1$,

$$L_{\rm max} \approx \frac{\mu\mathscr{U}_{\rm R}^2}{f_0}\left(\frac{4}{3\beta} - \frac{7}{18} + \frac{1}{45}\beta + \frac{1}{405}\beta^2\right), \tag{6.10}$$

explain the appearance of the first term.

In order to obtain the critical time $t_{\rm c}$, it is difficult to start directly from the Eq. (6.7) since the small quantity d appears both in the integration ranges and in the function of the form $\int_0^{X(d)} \mathscr{F}(\rho, d){\rm d}\rho$, it is essentially difficult to deal with.[4] However one could write the relevant expressions in terms of d and then introduce the perturbative structure of the X, i.e.,

$$t_{\rm c} = \sqrt{\frac{\mu}{k}}\int_0^{\pi/2}\frac{{\rm d}\phi}{\sqrt{1+\xi}} \approx \sqrt{\frac{\mu}{k}}\int_0^{\pi/2}{\rm d}\phi\left(1 - \frac{1}{2}\xi + \frac{3}{8}\xi^2\right), \tag{6.11}$$

[4] The structure is $\int_0^{X_0(1+\Phi_1 d+\Phi_2 d^2)}[\mathscr{F}(\rho,0) + \mathscr{F}'(\rho,0)d + 2^{-1}\mathscr{F}''(\rho,0)d^2]{\rm d}\rho$, where the dependence of the quantity X on d is emphasized.

where $\xi=[dX^{2n}/k(n+1)]\sum_{j=0}^{n}\sin^{2j}\phi$. The result to order d^2 is,

$$\frac{t_c}{t_c^0}\approx 1-\frac{X^{2n}}{\sqrt{\pi}k}\frac{\Gamma(n+3/2)}{\Gamma(n+2)}d+\frac{3g_n}{4\pi}\frac{X^{4n}}{k^2(n+1)^2}d^2$$
$$\approx 1-\frac{1}{\sqrt{\pi}}\frac{\Gamma(n+3/2)}{\Gamma(n+2)}\nu+\left(\frac{3g_n}{4\pi}\frac{1}{(n+1)^2}-\frac{2nk\Phi_1}{X_0^{2n}\sqrt{\pi}}\frac{\Gamma(n+3/2)}{\Gamma(n+2)}\right)\nu^2, \quad (6.12)$$

where Φ_1 is defined in (6.9), $\nu=X_0^{2n}d/k$, and $g_n=\int_0^{\pi/2}\mathrm{d}\phi(\sum_{j=0}^{n}\sin^{2j}\phi)^2$. In the second line, the X is expanded around X_0 to order d^2 (or equivalently to order ν^2.). If we treat the collision process is described by the Hooke's force with extra nonlinear interaction, then the effective spring constant could be constructed as to order ν^2,

$$k_{\rm eff}/k\approx 1+\frac{2}{\sqrt{\pi}}\frac{\Gamma(n+3/2)}{\Gamma(n+2)}\nu$$
$$+\left[\frac{3}{\pi}\left(\frac{\Gamma(n+3/2)}{\Gamma(n+2)}\right)^2-2\left(\frac{3g_n}{4\pi}\frac{1}{(n+1)^2}-\frac{2nk\Phi_1}{X_0^{2n}\sqrt{\pi}}\frac{\Gamma(n+3/2)}{\Gamma(n+2)}\right)\right]\nu^2. \quad (6.13)$$

Finally we give a mathematical example on using the "implicit form" of the perturbative method. Introducing the function $\chi(x)=1-x^\beta$ and its inverse function $\chi^{-1}(x)=(1-x)^{1/\beta}$ with $0\le x\le 1$, we denote $X(\beta)$ the zero of the function $\chi(x)-x$, with the former also the zero of the function $\chi^{-1}(x)-x$ and the function $f(x)=\chi(x)-\chi^{-1}(x)$ based on the relations among them. The aim of this example is to calculate the derivatives of the function $X(\beta)$ with respective to β and obtain the its value at $\beta=1$.

EXERCISE 14: Work out explicitly the expressions for $\mathrm{d}X/\mathrm{d}\beta$ and $\mathrm{d}^2X/\mathrm{d}\beta^2$ and then calculate the values of $X'(1)$ and $X''(1)$.

We could also obtain the needed derivatives via the "integration form" of the perturbative method as follows. First introducing the small quantity $\epsilon=0^+$, and then writing the β as $\beta=1+\epsilon$, one obtains

$$X(\beta)\approx X(1)+X'(1)\epsilon+\frac{1}{2}X''(1)\epsilon^2+\cdots,\ X^{(n)}(1)\equiv\left.\frac{\partial^n X}{\partial\beta^n}\right|_{\beta=1}, \quad (6.14)$$

$$X^\beta(\beta)\approx\left(\frac{1}{2}+X'(1)\epsilon+\frac{1}{2}X''(1)\epsilon^2+\cdots\right)^{1+\epsilon}\approx\frac{1}{2}+a_1\epsilon+a_2\epsilon^2+\cdots, \quad (6.15)$$

using the formula $a^{1+\epsilon}\approx a[1+\epsilon\log a+2^{-1}\epsilon^2\log^2 a]$ for small ϵ, where $a_1,a_2,\cdots$, are coefficients to be obtained order by order. Putting these expansions into the equation $X^\beta(\beta)+X(\beta)-1=0$ leads to

$$\left(a_1+X'(1)\right)\epsilon+\left(a_2+\frac{1}{2}X_1''(1)\right)\epsilon^2+\cdots=0. \quad (6.16)$$

By matching both sides of the equation, one easily obtains the $X'(1)$, and then a_2 (since a_2 depends on the a_1), and consequently $X''(1)$, etc. In principal the derivatives to all orders could be obtained in this manner, however the calculations become more complex as the order of the derivative increases. For example we have to order ϵ^2, we have $X(\beta)=2^{-1}+X'(1)\epsilon+2^{-1}X''(1)\epsilon^2$, and consequently,

$$X^\beta(\beta)=\frac{1}{2}+\left(X'(1)-\frac{\log 2}{2}\right)\epsilon$$
$$+\left(\frac{1}{2}X''(1)+X'(1)-X'(1)\log 2+\frac{\log^2 2}{4}\right)\epsilon^2, \quad (6.17)$$

from which one obtains $X'(1)=4^{-1}\log 2$ and $X''(1)=-4^{-1}\log 2$.

After some straightforward algebraic derivations, we obtain the general equation determining the first n derivatives $X^{(i)}(1)$ $(i=1\sim n)$,

$$\sum_{i=1}^{n}\frac{X^{(i)}(1)\epsilon^i}{i!}-\frac{1}{2}+\left[\frac{1}{2}+\sum_{i=1}^{n}\frac{(-\epsilon\log 2)^i}{2i!}\right]\times$$
$$\left[1+\sum_{j=1}^{n}\frac{\prod_{k=1}^{j}(2+\epsilon-k)}{j!}\left(\sum_{\ell=1}^{n}\frac{2X^{(\ell)}(1)\epsilon^\ell}{\ell!}\right)^j\right]=0. \quad (6.18)$$

In these applications the conceptual $\epsilon=0^+$ is very fundamental, although it does not matter what the exact value is. This is the intrinsic property of the integration form of the perturbation theories.

EXERCISE 15: Obtain the value of $X'''(1)$ using the implicit method.

EXERCISE 16: Denote $e_J=T+U_J$ as the single nucleon energy in asymmetric nucleonic matter with density $\rho=\sum_{J=\mathrm{n,p}}\rho_J$ and isospin asymmetry $\delta=(\rho_\mathrm{n}-\rho_\mathrm{p})/(\rho_\mathrm{n}+\rho_\mathrm{p})$, here $T=\mathbf{k}^2/2M$ is the kinetic energy with M being the static mass of nucleons, and $U_J=U_J(\rho,\delta,\mathbf{k})$ is the single nucleon potential. The density ρ_J and the momentum $\mathbf{k}$ is connected via the Fermi relation as $k_\mathrm{F}^J=(3\pi^2\rho_J)^{1/3}$ where k_F^J is called the Fermi momentum for nucleon J. The value of e_J at the Fermi momentum is called the chemical potential and is denoted as μ_J which is function of ρ and δ (and not of momentum $\mathbf{k}$). Thermodynamic relation tells that $\mu_J=\partial[\rho E(\rho,\delta)]/\partial\rho_J$ where $E(\rho,\delta)\approx E_0(\rho)+E_{\rm sym}(\rho)\delta^2+\cdots$ is the equation of state of the system and $E_{\rm sym}(\rho)$ is the symmetry energy. Derive using the integration form of the perturbative method the expressions for $E_{\rm sym}(\rho)$ and $L(\rho)=3\rho\mathrm{d}E_{\rm sym}(\rho)/\mathrm{d}\rho$ in terms of $U_0,U_{\rm sym}$ and $U_{\rm sym,2}$, with them defined via $U_J(\rho,\delta,\mathbf{k})=U_0(\rho,\mathbf{k})+U_{\rm sym}(\rho,\mathbf{k})\tau_3^J\delta+U_{\rm sym,2}(\rho,\mathbf{k})\delta^2$ under the convention $\tau_3^\mathrm{n}=+1$ (neutron) and $\tau_3^\mathrm{p}=-1$ (proton).

APPENDIX: DISCUSSION ON POLYNOMIALS

The polynomial is the most important form of the perturbative expansion in these calculations. Specifically, the function $f(x)$ could be expanded for example around $x=0$ to be $f(x)\approx f(0)+f'(0)x+2^{-1}f''(0)x^2+\cdots$. The polynomial expansion is very efficient in the sense that as long as the number of the model parameters is large enough, one could obtain almost any structure wanted. However, it is necessary to point that the starting point of the polynomial expansion is the existence of the small quantity (quantities), and neglecting this starting point in some cases maybe serious. It also indicates that the effectiveness of the perturbative expansion, or more generally the "effectiveness" of the effective theories.

For example, the single particle energy in special relativity is given by $E=(\mathbf{k}^2+M^2)^{1/2}$ where $\mathbf{k}$ is its momentum, and its non-relativistic approximation $E\approx\mathbf{k}^2/2M-\mathbf{k}^4/8M^3$ is effective when the quantity $|\mathbf{k}|/M$ is small. However when $|\mathbf{k}|\sim 2M$ the non-relativistic description itself is meaningless and the whole relativistic expression should be adopted. One of the equivalent treatments frequently used in the practical applications is to consider the coefficient in front of the $\mathbf{k}^4$ term as an effective parameter, i.e., $E\approx\mathbf{k}^2/2M+\alpha\mathbf{k}^4$, and determine the parameter α in some certain cases. Once the α is fixed by this way, the energy E with the parameter α could be used in other problems. We want to point out that the approach discussed here actually breaks the original structures of the theory, indicating it maybe not a good choice. Specifically, the $\mathbf{k}$ in the calculated quantities may already on the same order of $[2M|\alpha|]^{-1/2}$. It is also important to keep in mind that the real effective region is $|\mathbf{k}|\ll[2M|\alpha|]^{-1/2}$ instead of $|\mathbf{k}|\lesssim[2M|\alpha|]^{-1/2}$.

Finally, we briefly comment on the simple pendulum from the viewpoint of the polynomial expansion. The simple pendulum is actually a very useful elementary mechanical model, e.g., it could be treated as a microscopic gravity field. For the necessity we introduce the function $w(\chi)=\left|\partial f/\partial\chi\right|$. At small angle $\chi_{\max}\ll 1$, one has $w_{\rm small\,angle}(\chi)\approx mg=$ const., while generally $w(\chi)=mg\cos\chi$, which is not a constant and is depending on χ, or equivalently on the equation of motion. The function $w(\chi)$ could be expanded and thus $w(\chi)=mg(1-2^{-1}\chi^2+24^{-1}\chi^4-\cdots)$. If the angle is very small (e.g., near zero), the swing pattern of the ball is essential linear, which is in fact the geometrical meaning of the small-angle approximation. This feature is very similar as the one involved in general relativity, with the main idea that the space could be treated as flat at large scales, e.g., one could hardly figure out the curvature properties of the earth and could only treat it as flat effectively. The interesting and fruitful structures of the simple pendulum is originated from the non-Hooke's elasticity from the viewpoint of physics and the complexity of the function $\sin\chi$ (compared with χ) from the viewpoint of mathematics.

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Solutions to Exercises

1. We generally solve the equation (2.1) by setting

$$x(t) \approx x_0\left(1+\delta+c_2\delta^2+c_3\delta^3\right), \tag{s1}$$

where δ and c_2 and c_3 need to be determined. Naturally $\delta = kt$, thus at the third order, we have

$$x^n(t) \approx x_0^n\left[1+n\delta+\left(c_2 n+\frac{1}{2}n(n-1)\right)\delta^2 + \left(c_3 n + c_2 n(n-1)+\frac{1}{6}n(n-1)(n-2)\right)\delta^3\right], \tag{s2}$$

for the left hand side of Eq. (2.1). In order to expand the right hand side to the third order, we notice that $\delta = kt \sim t$, thus

$$\Omega+\frac{tx}{\Lambda} \approx \Omega+\frac{tx_0}{\Lambda}\left(1+\delta+c_2\delta^2+c_3\delta^3\right) \approx \Omega+\frac{\delta x_0}{k\Lambda}\left(1+\delta+c_2\delta^2\right), \tag{s3}$$

where $t=\delta/k$ is used. Compare the equations (s2) and (s3), one has

(a) For the zeroth-order theory, we have $x_0 = \Omega^{1/n}$.

(b) For the first-order theory, we obtain $k = x_0^{1-n}/n\Lambda = \Omega^{1/n-1}/n\Lambda$.

(c) Similarly for the second-order and the third-order theories, one can obtain

$$c_2 = \frac{3-n}{2}, \quad c_3 = \frac{n^2-6n+8}{3}, \tag{s4}$$

thus

$$x(t) \approx x_0\left[1+\delta(t)-\frac{n-3}{2}\delta^2(t)+\frac{n^2-6n+8}{3}\delta^3(t)\right], \tag{s5}$$

for small t approximation.

The conditions for the first-, second- and third- order theories are given respectively by,

$$\delta(t) \ll 1, \quad \delta(t) \ll \frac{2}{n-3}, \quad \delta(t) \ll \frac{3}{2}\frac{n-3}{n^2-6n+8}, \tag{s6}$$

where $n \geq 5$. Denote $f_1 = 1, f_2 = 2/(n-3)$ and $f_3 = 3(n-3)/2(n^2-6n+8)$, one can show that $f_2 \leq f_1$, and $f_3 \leq f_2$, as

$$\frac{f_3}{f_2} = \frac{3}{4}\left(1+\frac{1}{n^2-6n+8}\right) \leq \frac{3}{4}\left(1+\frac{1}{3}\right) = 1, \tag{s7}$$

where the equality holds when $n = 5$. These results show that as the expansion order increases, the effective perturbative region shrinks. However the expansion becomes better, as discussed in the main context and shown in Fig. s1. On the other hand, for the asymptotic perturbation, one already has $x_\infty(t) = (t/\Lambda)^{n/(n-1)}$. By setting

$$x(t) \approx x_\infty(t)\left(1+\phi+d_2\phi^2+d_3\phi^3\right), \tag{s8}$$

one obtains

$$\text{LHS} = \left(\frac{t}{\Lambda}\right)^{\frac{n}{n-1}} + n\phi\left(\frac{t}{\Lambda}\right)^{\frac{n}{n-1}} + \left(\frac{t}{\Lambda}\right)^{\frac{n}{n-1}}\left[\left(nd_2+\frac{1}{2}n(n-1)\right)\phi^2 + \left(nd_3+d_2 n(n-1)+\frac{1}{6}n(n-1)(n-2)\right)\phi^3\right], \tag{s9}$$

$$\text{RHS} = \Omega + \left(\frac{t}{\Lambda}\right)^{\frac{n}{n-1}} + \phi\left(\frac{t}{\Lambda}\right)^{\frac{n}{n-1}} + \left(\frac{t}{\Lambda}\right)^{\frac{n}{n-1}}\left(d_2\phi^2+d_3\phi^3\right). \tag{s10}$$

Comparing order by order gives

$$\phi = \frac{\Omega}{n-1}\left(\frac{\Lambda}{t}\right)^{\frac{n}{n-1}}, \quad d_2 = -\frac{1}{2}n, \quad d_3 = \frac{1}{3}n(n+1). \tag{s11}$$

Thus

$$x(t) \approx x_\infty(t)\left[1+\phi(t)-\frac{1}{2}n\phi^2(t)+\frac{1}{3}n(n+1)\phi^3(t)\right]. \tag{s12}$$

The conditions for the first-, second- and third- order theories are given respectively by,

$$\phi(t) \ll 1, \quad \phi(t) \ll \frac{2}{n}, \quad \phi(t) \ll \frac{3}{2(n+1)}, \tag{s13}$$

where $n \geq 5$. Denote $g_1 = 1, g_2 = 2/n$ and $g_3 = 3/2(n+1)$, one can show that $g_2 < g_1$, and $g_3 < g_2$, as

$$\frac{g_3}{g_2} = \frac{3n}{4(n+1)} \leq \frac{5}{8}. \tag{s14}$$

See Fig. s1. If one still sets $x^{\text{3rd}}_{\text{small}-t}(t) \approx x^{\text{3rd}}_{\text{large}-t}(t)$ then $0 \lesssim t \lesssim 1.48$.

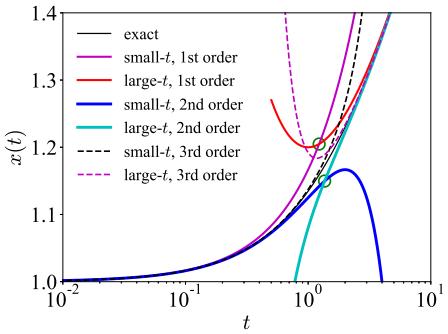

Fig. s1: Solution of Eq. (2.1) where $n = 6, \Lambda = \Omega = 1$, to third order.

2. This is the direct generalization of EXERCISE 1. For the small t limit, one still has $x_0 = \Omega^{1/n}$, and by expanding the left and right hand sides of the equation, we obtain

$$\text{LHS} \approx x_0^n\left[1+n\delta+\left(c_2 n+\frac{1}{2}n(n-1)\right)\delta^2 + \left(c_3 n + c_2 n(n-1)+\frac{1}{6}n(n-1)(n-2)\right)\delta^3\right], \tag{s15}$$

$$\text{RHS} \approx \Omega + \frac{x_0^m\delta}{k\Lambda}\left[1+m\delta+\left(c_2 m+\frac{1}{2}m(m-1)\right)\delta^2\right], \tag{s16}$$

where the left hand side is just (s2) and the coefficient k is introduced through $\delta = kt$. Comparing them order by order gives $\delta(t) = (1/n\Lambda)\Omega^{m/n-1}$, and

$$c_2 = \frac{1}{2}(2m-n+1), \tag{s17}$$

$$c_3 = \frac{1}{2}\left((2m-n+1)(m-n+1)+m(m-1)-\frac{1}{3}n(n-1)(n-2)\right), \tag{s18}$$

thus

$$x(t) \approx \Omega^{1/n}\left[1+\delta(t)+\frac{2m-n+1}{2}\delta^2(t) + \frac{1}{2}\left((2m-n+1)(m-n+1)+m(m-1)-\frac{1}{3}n(n-1)(n-2)\right)\delta^3(t)\right]. \tag{s19}$$

On the opposite limit, i.e., large t, we have now $x_\infty(t) = (t/\Lambda)^{1/(n-m)}$, and

$$\text{LHS} = \left(\frac{t}{\Lambda}\right)^{\frac{n}{n-m}} + n\phi\left(\frac{t}{\Lambda}\right)^{\frac{n}{n-m}} + \left(\frac{t}{\Lambda}\right)^{\frac{n}{n-m}}\left[\left(nd_2+\frac{1}{2}n(n-1)\right)\phi^2 + \left(nd_3+d_2 n(n-1)+\frac{1}{6}n(n-1)(n-2)\right)\phi^3\right], \tag{s20}$$

$$\text{RHS} = \Omega + \left(\frac{t}{\Lambda}\right)^{\frac{n}{n-m}} + m\phi\left(\frac{t}{\Lambda}\right)^{\frac{n}{n-m}} + \left(\frac{t}{\Lambda}\right)^{\frac{n}{n-m}}\left[\left(md_2+\frac{1}{2}m(m-1)\right)\phi^2 + \left(md_3+d_2 m(m-1)+\frac{1}{6}m(m-1)(m-2)\right)\phi^3\right]. \tag{s21}$$

From them, we have

$$\phi = \frac{\Omega}{n-m}\left(\frac{\Lambda}{t}\right)^{\frac{n}{n-m}}, \tag{s22}$$

and

$$d_2 = -\frac{n+m-1}{2}, \tag{s23}$$

$$d_3 = \frac{1}{2}(n+m-1)^2 - \frac{1}{6}\left(n^2+nm+m^2-3(n+m)+2\right), \tag{s24}$$

and consequently,

$$x(t) \approx \left(\frac{t}{\Lambda}\right)^{\frac{1}{n-m}}\left[1+\phi(t)-\frac{n+m-1}{2}\phi^2(t) + \left(\frac{1}{2}(n+m-1)^2 - \frac{1}{6}\left(n^2+nm+m^2-3(n+m)+2\right)\right)\phi^3(t)\right]. \tag{s25}$$

3. Denote the intersection between the lines $x^{\rm 1st}_{{\rm small}-t}(t)$ and $x^{\rm 1st}_{{\rm large}-t}(t)$ as $x_{\rm c}^{(1)}$, then the absolute value of the area S_1 formed by the exact solution, the first-order curves both for small t and large t is given by

$$S_1 = \left|\int_0^{x_{\rm c}^{(1)}} {\rm d}t\, x^{\rm 1st}_{{\rm small}-t}(t) + \int_{x_{\rm c}^{(1)}}^{\infty} {\rm d}t\, x^{\rm 1st}_{{\rm large}-t}(t) - \int_0^{\infty} {\rm d}t\, x_{\rm exact}(t)\right|, \tag{s26}$$

here the first two integrations could be evaluated using the analytical expressions. However the last one can only be evaluated by some numerical algorithms, e.g., the Simpson's method. Since the difference between the large-t and the exact curves are small at $t \gtrsim 10$, the "∞" in the above integrations could be safely replaced by a large value, e.g., 10. Consequently, one obtains

$$S_1 \approx 0.0757. \tag{s27}$$

Very similarly,

$$S_2 = \left|\int_0^{x_{\rm c}^{(2)}} {\rm d}t\, x^{\rm 2nd}_{{\rm small}-t}(t) + \int_{x_{\rm c}^{(2)}}^{\infty} {\rm d}t\, x^{\rm 2nd}_{{\rm large}-t}(t) - \int_0^{\infty} {\rm d}t\, x_{\rm exact}(t)\right|, \tag{s28}$$

$$S_3 = \left|\int_0^{x_{\rm c}^{(3)}} {\rm d}t\, x^{\rm 3rd}_{{\rm small}-t}(t) + \int_{x_{\rm c}^{(3)}}^{\infty} {\rm d}t\, x^{\rm 3rd}_{{\rm large}-t}(t) - \int_0^{\infty} {\rm d}t\, x_{\rm exact}(t)\right|, \tag{s29}$$

where $x_{\rm c}^{(2,3)}$ is the intersection between the lines $x^{\rm 2nd,3rd}_{{\rm small}-t}(t)$ and $x^{\rm 2nd,3rd}_{{\rm large}-t}(t)$. Consequently, $S_2 \approx 0.0210$ and $S_3 \approx 0.0112$. See Fig. s2 for the full curves by linking the small-t and the large-t solutions. From this figure it is shown that the third order result gives very close prediction from the exact theory.

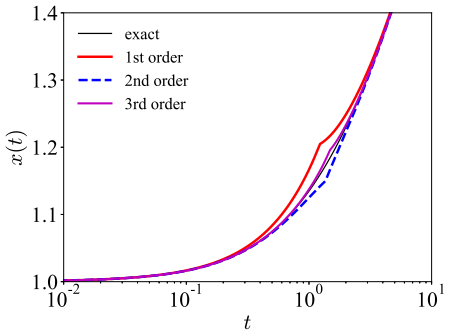

Fig. s2: Solution of Eq. (2.1) where $n = 6, \Lambda = \Omega = 1$ using the full curves.

4. Assume that the ball moves under the initial condition $\chi = 0$ and $v(0) = l\dot\chi(0) = V$. The equation of motion of the ball is given by $\ddot x + g\sin\chi = 0$, or $\ddot\chi + (g/l)\sin\chi = 0$, since $x = \chi l$. For small angle, we have $\sin\chi \approx \chi$, the equation of motion becomes $\ddot\chi + (g/l)\chi = 0$. The general solution of this equation is $\chi(t) = a\sin\omega t + b$ where $\omega^2 = g/l$. The constants a and b could be determined via the initial condition as $a = V/\sqrt{gl}$ and $b = 0$, i.e., $\chi(t) = (V/\sqrt{gl})\sin\omega t$. Consequently, the period of the motion is obtained namely $T = 2\pi/\omega = 2\pi\sqrt{l/g}$.

5. The indices μ, ν and σ could not be any real numbers since any quantity f should be a rational composition of the quantities with mass, length and time as their basic units, through linear equations.

6. The solution for the equation $\ddot\chi + \phi\chi = 0$ under the corresponding initial conditions is $\chi_0(t) \equiv (C/\sqrt{\phi})\sin\sqrt{\phi}t$. When the term $-6^{-1}\phi\chi^3$ is included, we denote the general solution as $\chi(t) = \chi_0(t)[1+\Delta(t)]$ where $\Delta(t)$ is the perturbation owning the property:

$$|\Delta(t)| \ll 1, \tag{s30}$$

and moreover $\Delta(t)$ is assumed to be a slow-varying function. In this sense, the equation of motion becomes

$$\ddot\Delta(t) + \frac{2\dot\chi_0(t)}{\chi_0(t)}\dot\Delta(t) - \frac{1}{2}\phi\chi_0^2(t)\Delta(t) \approx \frac{1}{6}\phi\chi_0^2(t), \tag{s31}$$

where the equation of motion for $\chi_0(t)$, i.e., $\ddot\chi_0(t) + \phi\chi_0(t) = 0$ is used when deriving the above equation. Since $\Delta(t)$ is a slow-varying function, the first term on the left hand side will be neglected in the following study. Moreover, according to (s30), one can neglect the third term on the left hand side when compared with the right hand side. Consequently, one has

$$\dot\Delta(t) \approx \frac{1}{12}\frac{\phi\chi_0^3(t)}{\dot\chi_0(t)}, \tag{s32}$$

and this is the equation we want to solve. The result is

$$\Delta(t) = -\frac{C^2}{12\phi}\left(\frac{1}{2}\sin^2\sqrt{\phi}t + \log\left|\cos\sqrt{\phi}t\right|\right), \tag{s33}$$

using the general formula,

$$\int\frac{\sin^2 ax}{\cos ax}{\rm d}x = -\frac{1}{2a}\sin^2 ax - \frac{1}{a}\log|\cos ax| + {\rm const.} \tag{s34}$$

Consequently,

$$\chi(t) \approx \frac{C}{\sqrt{\phi}}\sin\sqrt{\phi}t\left[1 - \frac{C^2}{12\phi}\left(\frac{1}{2}\sin^2\sqrt{\phi}t + \log\left|\cos\sqrt{\phi}t\right|\right)\right]. \tag{s35}$$

Fig. s3 gives an example where $C = 1$ and $\phi = 1.2$. The breakdown at $\sqrt{\phi}t = \pi/2$ is due to the second term of (s33). The exact (numerical) result could be obtained by, e.g., the Euler's algorithm or the Runge–Kutta method, here we adopt the 4th order Runge–Kutta algorithm.

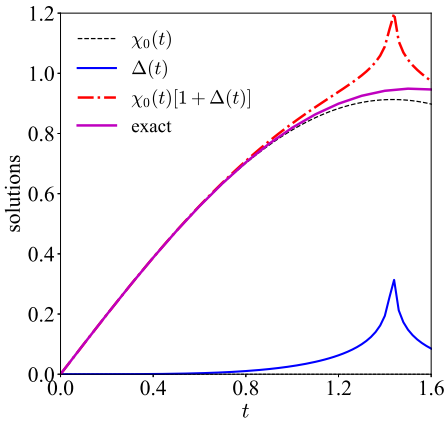

Fig. s3: Example of EXERCISE 6 with $C = 1$ and $\phi = 1.2$.

7. From the formula for the root, one can obtain,

$$x^* = \frac{b}{2a}\left(\sqrt{1-k}-1\right) \approx -\frac{b}{2a}\cdot\frac{k}{2}\left(1+\frac{k}{4}\right), \tag{s36}$$

where $k = 4ac/b^2$. The above result is

$$x^* \approx -\frac{c}{b}\left(1+\frac{ac}{b^2}\right). \tag{s37}$$

On the other hand, if one starts from the root for the linear equation, i.e., $x_0 = -c/b$, of $bx+c=0$, and then sets $x \approx x_0(1+p)$, the correction p is easy to find as through putting it into $ax^2+bx+c=0$ under the assumption that ac/b^2 is small,

$$p = \frac{ac^2}{b}\Big/\left(c-\frac{2ac^2}{b^2}\right) \approx \frac{ac}{b^2}, \tag{s38}$$

which is the same as the one obtained already.

8. For the simple pendulum with small swing angle with $\chi \lesssim 5^\circ$, the kinetic energy, the potential energy and the total energy of the pendulum are given by $K(\dot\chi) = ml^2\dot\chi^2/2, U(\chi) = mgl\chi^2/2, E = mgl(1-\cos\chi_{\max}) \approx mgl\chi_{\max}^2/2$(the zero point of the potential energy is selected at the bottom of the oscillation), consequently $f = ml^2/2, g = mgl/2$, and one has $\omega = (g/l)^{1/2}, T = 2\pi(l/g)^{1/2}$. If the zero point of the potential energy is selected at the top of the oscillation, $U(\chi) = -mgl\cos\chi \approx mgl\chi^2/2 - mgl, C = -mgl, E = -mgl\cos\chi_{\max} \approx mgl\chi_{\max}^2/2 - mgl$, and $E - C = mgl\chi_{\max}^2/2$, indicating the relevant results are independent of the selection of the zero point. Next, denote χ_1 and χ_2 the angles of the oscillation of the two pendulums, l the length and m the mass. Then the kinetic energies and the potential energies are given,

$$K_1(\dot\chi_1) = \frac{1}{2}ml^2\dot\chi_1^2, \tag{s39}$$

$$K_2(\dot\chi_1,\dot\chi_2) = \frac{1}{2}ml^2\left(\dot\chi_1+\dot\chi_2\right)^2, \tag{s40}$$

$$U_1(\chi_1) = -mgl\cos\chi_1 \approx -mgl\left(1-\frac{1}{2}\chi_1^2\right), \tag{s41}$$

$$U_2(\chi_1,\chi_2) = -mgl\left(\cos\chi_1+\cos\chi_2\right) \approx -mgl\left(2-\frac{1}{2}\chi_1^2-\frac{1}{2}\chi_2^2\right). \tag{s42}$$

Thus, the total energy is obtained,

$$E(\chi_1,\chi_2,\dot\chi_1,\dot\chi_2) \approx \frac{1}{2}ml^2\left(2\dot\chi_1^2+2\dot\chi_1\dot\chi_2+\dot\chi_2^2\right) - mgl\left(3-\chi_1^2-\frac{1}{2}\chi_2^2\right). \tag{s43}$$

In order to remove the cross term $\dot\chi_1\dot\chi_2$, we introduce the new coordinates $\phi = \sqrt{2}\chi_1+\chi_2$ and $\varphi = \sqrt{2}\chi_1-\chi_2$, then the total energy becomes

$$E(\phi,\varphi,\dot\phi,\dot\varphi) = \frac{2+\sqrt{2}}{8}ml^2\dot\phi^2+\frac{1}{4}mgl\dot\phi^2 + \frac{2-\sqrt{2}}{8}ml^2\dot\varphi^2+\frac{1}{4}mgl\dot\varphi^2-3mgl, \tag{s44}$$

from which one reads immediately $f_\phi = (2+\sqrt{2})ml^2/8, g_\phi = mgl/4$ and $f_\varphi = (2-\sqrt{2})ml^2/8, g_\varphi = mgl/4$. Consequently,

$$\omega_{\phi/\varphi} = \sqrt{\frac{g_{\phi/\varphi}}{f_{\phi/\varphi}}} = \sqrt{\frac{(2\pm\sqrt{2})g}{l}}. \tag{s45}$$

9. For the extra force $f^\delta(x) = ax^3$, one has generally for the period,

$$T = 4\sqrt{\frac{m}{k}}\int_0^{\pi/2}\mathrm{d}\phi\frac{1}{\sqrt{1+\dfrac{1+\sin^2\phi}{2}\sigma}}, \quad \sigma = \frac{aX^2}{k}, \tag{s46}$$

which could be expanded around $\sigma = 0$ to obtain,

$$T \approx 2\pi\sqrt{\frac{m}{k}}\times\left(1-\frac{3}{8}\sigma+\frac{57}{256}\sigma^2-\frac{315}{2048}\sigma^3+\frac{30345}{262144}\sigma^4\right). \tag{s47}$$

On the other hand, by using $k_{\rm eff} \approx k(1+s_1\sigma+s_2\sigma^2+s_3\sigma^3+s_4\sigma^4)$, one has the expansion,

$$T \approx 2\pi\sqrt{\frac{m}{k}}\times\left[1-\frac{1}{2}s_1\sigma+\left(\frac{3}{8}s_1^2-\frac{1}{2}s_2\right)\sigma^2 + \left(-\frac{5}{16}s_1^3+\frac{3}{4}s_1s_2-\frac{1}{2}s_3\right)\sigma^3 + \left(\frac{35}{128}s_1^4-\frac{15}{16}s_1^2s_2+\frac{3}{4}s_1s_3+\frac{3}{8}s_2^2-\frac{1}{2}s_4\right)\sigma^4\right], \tag{s48}$$

via $T = 2\pi\sqrt{m/k_{\rm eff}}$. Comparing (s47) and (s48) gives

$$s_1 = \frac{3}{4},\ s_2 = -\frac{3}{128},\ s_3 = \frac{9}{512},\ s_4 = -\frac{1779}{131072}. \tag{s49}$$

Consequently, the effective Hooke's constant to σ^4 is given as

$$k_{\rm eff} \approx k\times\left(1+\frac{3}{4}\sigma-\frac{3}{128}\sigma^2+\frac{9}{512}\sigma^3-\frac{1779}{131072}\sigma^4\right). \tag{s50}$$

10. Expanding the potential $\widetilde{U}(\chi)$ to order χ^5 gives,

$$\widetilde{U}(\chi) \approx -\sqrt{\frac{27}{50}}(1-\theta)^{5/2}\chi+\frac{1}{4}(1-\theta)(1+3\theta)\chi^2+\theta[(1-\theta)/6]^{1/2}\chi^3 + \frac{1}{24}(3\theta-2)\chi^4-\frac{1}{120}\sqrt{6(1-\theta)}\chi^5+\widetilde{U}_0, \tag{s51}$$

the last two terms of the above potential are the 4th-order and the 5th-order self-interactions for the χ field, respectively.

11. One of the most simple algorithms for finding β is via iteration,

$$\beta^{(i+1)} = \frac{1}{\theta}\sin\beta^{(i)}, \tag{s52}$$

through an initial $\beta^{(0)}$. In order to find the first-order correction to the solution $B = \sqrt{6(1-\theta)}$, set $\beta = B(1+\Delta)$, where Δ could be found through the equation $\theta\beta \approx \beta - 6^{-1}\beta^3 + 120^{-1}\beta^5$, the result is

$$\Delta \approx \frac{3}{4}\frac{1-\theta}{2+3\theta}. \tag{s53}$$

Thus

$$\beta \approx [6(1-\theta)]^{1/2}\cdot\left(1+\frac{3}{4}\frac{1-\theta}{2+3\theta}\right). \tag{s54}$$

See Fig. s4 for an example where $\beta^{(0)} = 0.1$ and $\theta = 0.8$.

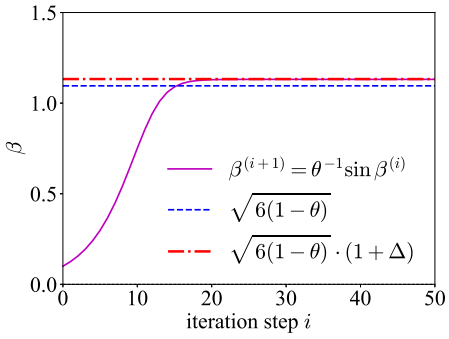

Fig. s4: Solution of $\theta\beta = \sin\beta$ with its two approximations.

12. The period of the motion is given by

$$T = \sqrt{\frac{2m}{U_0}}\times\int_{X_1}^{X_2}\frac{\mathrm{d}x}{\sqrt{1-\alpha_1x^3-\alpha_2x^2}}, \tag{s55}$$

where $\alpha_1=(\nu-1)/\nu^2X_1^3$ and $\alpha_2=(\nu^2-\nu+1)/\nu^2X_1^2$. The integrand could be expanded as

$$\frac{1}{\sqrt{1-\alpha_1x^3-\alpha_2x^2}}\approx\left(1-\left(\frac{x}{X_2}\right)^2\right)^{-1/2}\left[1+\frac{x^2}{2(X_2+x)X_2}f+\frac{(4X_2^2+4X_2x+3x^2)x^2}{8(X_2+x)^2X_2^2}f^2\right], \tag{s56}$$

where $f=\nu-1$. The zeroth-order term gives $T_0=\sqrt{2m/U_0}\times X_2\pi$, and now $n_1=0$ (since $\nu=1$). Thus $U=U_0x^2/X_1^2=U_0x^2/X_2^2=2^{-1}kx^2$, leading to the effective Hooke's constant as $k=2U_0/X_2^2$. By putting this k into the general formula $T=2\pi\sqrt{m/k}$ gives $\pi X_2\sqrt{2m/U_0}$. In order to obtain the corrections, we set $X_1=-X_2/\nu\approx-X_2(1-f+f^2)$ and do the integration order by order, the result is given as,

$$T\approx\pi X_2\sqrt{\frac{2m}{U_0}}\times\left(1+\frac{5\sqrt{2}f^{1/2}}{8\pi}-\frac{f}{2}+\frac{13\sqrt{2}f^{3/2}}{32\pi}+\frac{13f^2}{16}\right). \tag{s57}$$

Here X_2 should be determined by the potential $U(X_2)=U_0$ by considering $X_2>0$ and the potential is expanded to f^2, thus this X_2 has the different meaning from that in the zeroth-order approximation. In order to obtain the critical range for ν, let's do as follows: Take the derivative of $U(x)$ with respect to x and set it to be zero, the resulted value for x is denoted as X, then take the value of $U(X)$ and demand it to be smaller than U_0, the result for ν is consequently obtained as $\nu\le2$, i.e., $0\le f\le1$. For example, taking $\nu=1.5$ (1.2) leads to the relative value of 91.1% (96.6%) from the above approximation, compared with the exact period.

13. For ball 1, one has $\int_{u_1}^{v_1}\mathrm{d}v=\int_0^t a_1\mathrm{d}t$ where u_1 is the initial velocity and a_1 is the acceleration of the first ball, i.e.,

$$a_1=-\frac{f_0}{m_1}\left(e^{\beta\bar{t}}-1\right),\ \ \bar{t}=t/t_\mathrm{c}. \tag{s58}$$

Integrating the equation of motion gives the velocity

$$v_1(t)=u_1+\frac{f_0}{m_1}\left[t-\frac{t_\mathrm{c}}{\beta}\left(e^{\beta\bar{t}}-1\right)\right]. \tag{s59}$$

By equaling $v_1(t_\mathrm{c})$ to $V_\mathrm{c}=(m_1u_1+m_2u_2)/(m_1+m_2)$, one obtains the critical time t_c, as

$$t_\mathrm{c}=\frac{\mu\mathscr{U}_\mathrm{R}}{f_0}\frac{\beta}{e^\beta-\beta-1}, \tag{s60}$$

where $\mathscr{U}_\mathrm{R}$ is the initial relative velocity between the two balls, i.e., $\mathscr{U}_\mathrm{R}=u_1-u_2$, and μ is their reduced mass. Similarly, the velocity of the second ball is given by,

$$v_2(t)=u_2-\frac{f_0}{m_2}\left[t-\frac{t_\mathrm{c}}{\beta}\left(e^{\beta\bar{t}}-1\right)\right]. \tag{s61}$$

Consequently, one obtains the relative velocity $V(t)=U_\mathrm{R}=v_1(t)-v_2(t)$,

$$V(t)=\mathscr{U}_\mathrm{R}+\frac{f_0}{\mu}\left[t-\frac{t_\mathrm{c}}{\beta}\left(e^{\beta\bar{t}}-1\right)\right] \tag{s62}$$

The compression distance is $L(t)=\int_0^tV(t)\mathrm{d}t$, and thus

$$L_\mathrm{max}=L(t_\mathrm{c})=\frac{\mu\mathscr{U}_\mathrm{R}^2}{f_0}\frac{\beta}{e^\beta-\beta-1}\left(1-\frac{1}{\beta}+\frac{1}{2}\frac{\beta}{e^\beta-\beta-1}\right). \tag{s63}$$

For small $\beta\ll1$, the maximum distance is

$$L_\mathrm{max}\approx\frac{\mu\mathscr{U}_\mathrm{R}^2}{f_0}\left(\frac{4}{3\beta}-\frac{7}{18}+\frac{1}{45}\beta+\frac{1}{405}\beta^2\right). \tag{s64}$$

As $\beta\to0$, the critical time $t_\mathrm{c}\approx2\mu\mathscr{U}_\mathrm{R}/f_0\beta$ becomes very large, indicating that the collision process is long. Similarly, the interaction force under $\beta\to0$ becomes $F\approx-f_0\beta\bar{t}\to0$, indicating that the model is unreasonable if β is very small. Naturally, we have the condition that $L_\mathrm{max}\le2d$, with d the diameter of the ball, i.e.,

$$\frac{\mu\mathscr{U}_\mathrm{R}^2}{f_0}\frac{\beta}{e^\beta-\beta-1}\left(1-\frac{1}{\beta}+\frac{1}{2}\frac{\beta}{e^\beta-\beta-1}\right)\le2d, \tag{s65}$$

this gives a self-consistent constrain on the β parameter if $\mu,\mathscr{U}_\mathrm{R}$ and f_0 are known. For example, by keeping the lowest order term $4/3\beta$, we obtain the condition for β as

$$\beta\ge\beta_\mathrm{cr}\equiv\frac{2\mu\mathscr{U}_\mathrm{R}^2}{3df_0}. \tag{s66}$$

Naturally, $\beta_\mathrm{cr}\ll1$, should be fulfilled, i.e., $2\mu\mathscr{U}_\mathrm{R}^2/3df_0\ll1$.

14. The most direct approach to the problem is calculating the derivative of the equation $X^\beta+X-1=0$ with respect to β, leading to

$$\frac{\mathrm{d}X}{\mathrm{d}\beta}=-\frac{X^\beta\log X}{1+\beta X^{\beta-1}}, \tag{s67}$$

and consequently one obtains $\mathrm{d}X/\mathrm{d}\beta(\beta=1)=4^{-1}\log2$. Similarly,

$$\frac{\mathrm{d}^2X}{\mathrm{d}\beta^2}=\frac{X^\beta\log^2X(\beta X^{2\beta-2}-1)}{(1+\beta X^{\beta-1})^3}+\frac{2X^{2\beta-1}\log X}{(1+\beta X^{\beta-1})^2}, \tag{s68}$$

and thus $\mathrm{d}^2X/\mathrm{d}\beta^2(\beta=1)=-4^{-1}\log2$.

15. In the implicit method, we expand $X(\beta)=X(1+\epsilon)$ to order ϵ^3 as,

$$X(1+\epsilon)\approx a+b\epsilon+c\epsilon^2+d\epsilon^3, \tag{s69}$$

where

$$a=X(1),\ \ b=X'(1),\ \ c=\frac{1}{2}X''(1),\ \ d=\frac{1}{6}X'''(1). \tag{s70}$$

The expansion of $X^\beta(\beta)=X^{1+\epsilon}(1+\epsilon)$ is thus $X^\beta(\beta)\approx(a+b\epsilon+c\epsilon^2+d\epsilon^3)^{1+\epsilon}$, or

$$X^\beta(\beta)\approx a+(a\log a+b)\epsilon+\left(\frac{1}{2}a\log^2a+b\log a+b+c\right)\epsilon^2+\left[d+\frac{1}{6a}\left(a^2\log^3a+3ab\log^2a+6ab\log a+6ac\log a+6ac+6ad+3b^2\right)\right]\epsilon^3, \tag{s71}$$

where the formula $a^{1+\epsilon}\approx a[1+\epsilon\log a+2^{-1}\epsilon^2\log^2a+6^{-1}\epsilon^3\log^3a]$ is used. By comparing both sides of the equation $X^\beta(\beta)+X(\beta)=1$ or $X^{1+\epsilon}(1+\epsilon)+X(1+\epsilon)=1$, one obtains $a=X(1)=2^{-1}$, and

$$X'(1)=b=\frac{1}{4}\log2\approx0.1733, \tag{s72}$$

$$X''(1)=2c=-\frac{1}{4}\log2\approx-0.1733, \tag{s73}$$

$$X'''(1)=6d=\frac{3}{8}\log2+\frac{3}{16}\log^22-\frac{1}{8}\log^32\approx0.3084. \tag{s74}$$

16. The expressions for $E_\mathrm{sym}(\rho)$ and $L(\rho)$ are given as*

$$E_\mathrm{sym}(\rho)=\frac{k_\mathrm{F}^2}{6M}+\frac{k_\mathrm{F}}{6}\frac{\partial U_0}{\partial|\mathbf{k}|}\bigg|_{|\mathbf{k}|=k_\mathrm{F}}+\frac{1}{2}U_\mathrm{sym}(\rho,k_\mathrm{F}), \tag{s75}$$

$$L(\rho)=\frac{k_\mathrm{F}^2}{3M}+\left[\frac{k_\mathrm{F}^2}{6}\frac{\partial^2U_0}{\partial|\mathbf{k}|^2}+\frac{k_\mathrm{F}}{6}\frac{\partial U_0}{\partial|\mathbf{k}|}\right]_{|\mathbf{k}|=k_\mathrm{F}}+k_\mathrm{F}\frac{\partial U_\mathrm{sym}}{\partial|\mathbf{k}|}\bigg|_{|\mathbf{k}|=k_\mathrm{F}}+\frac{3}{2}U_\mathrm{sym}(\rho,k_\mathrm{F})+3U_{\mathrm{sym},2}(\rho,k_\mathrm{F}). \tag{s76}$$

*See, e.g., R. Chen *et al.*, *Single-nucleon potential decomposition of the nuclear symmetry energy*, Phys. Rev. C **85**, 024305 (2012), section II, for some details. The method is based on the Hugenholtz–Van Hove theorem.