Lecture 10

Large-scale Matrix Computations and 3D Reconstruction

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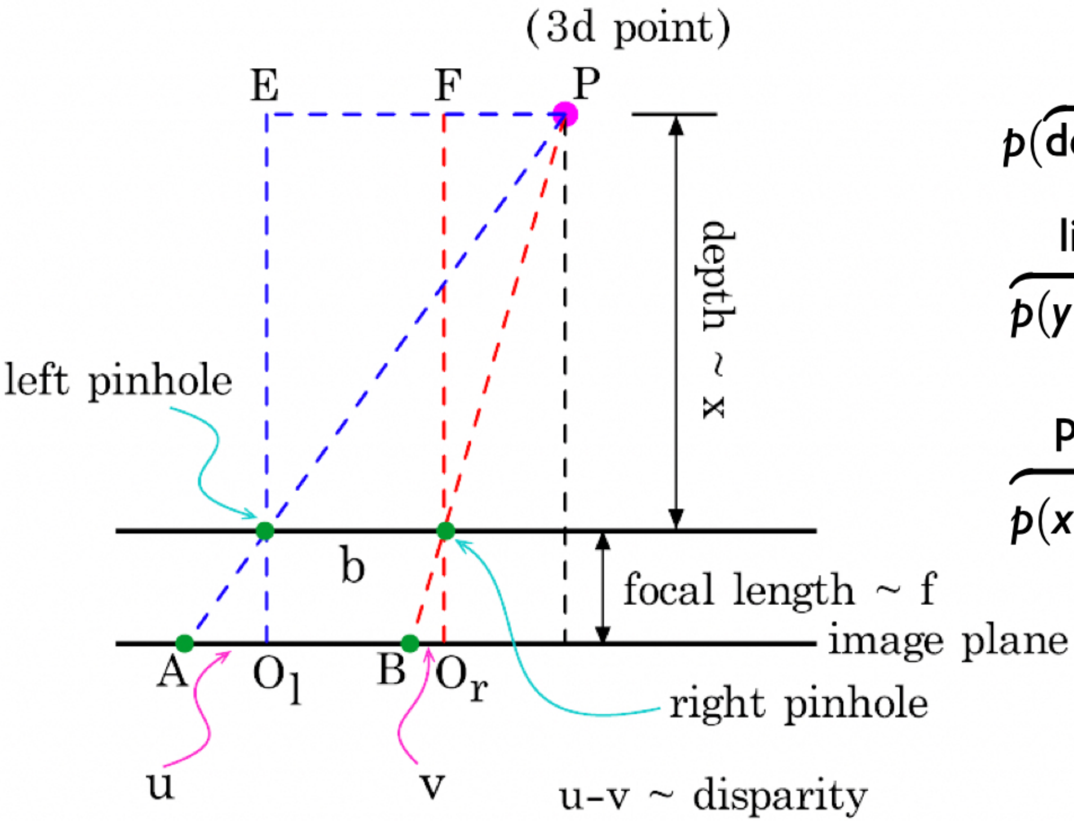
Topics of this lecture:

- depth and disparity $y = fb/x + n$
- epipolar constraint and fundamental matrix $\langle \mathbf{x}' | \mathbf{F} | \mathbf{x} \rangle = \mathbf{x}'^{\top} \mathbf{F} \mathbf{x} = 0$
- least-squares revisited "$\min_{\mathbf{f}} \|\mathbf{A}\mathbf{f}\|^2$"≈"$\mathbf{A}\mathbf{f} = \vec{0}$"
- triangulation and re-projection error $e(\mathbf{X}) = \sum_{j=1,2} r_j^2(\mathbf{X}),\ r_j^2(\mathbf{X}) = \|\mathbf{q}_j - \mathbf{X}\|^2$
- 3D reconstruction and SLAM $(\mathbf{X}^{(i)}, \mathbf{c}_j) \to \mathbf{x}_j^{(i)}$
- bundle adjustment $\mathbf{J}^{\top}\mathbf{J}\vec{\delta} = \mathbf{J}^{\top}\vec{\epsilon}$

Example: depth from disparity



$$y = fb/x + n$$

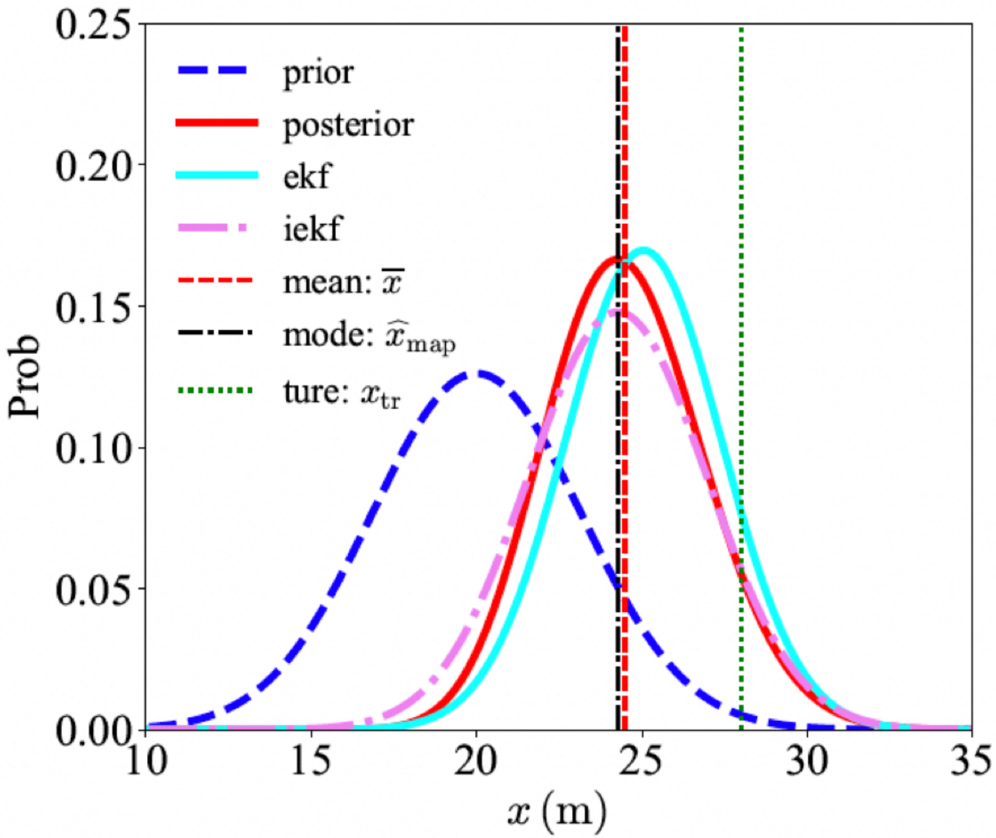
$$p(\overbrace{\text{depth}}^{x} \mid \overbrace{\text{disparity}}^{y}) = p(y|x)p(x) / \int dx\, p(y|x)p(x)$$

$$\overbrace{p(y|x) = \mathcal{N}(fb/x, Q)}^{\text{likelihood for } x} = \frac{1}{\sqrt{2\pi Q}} \exp\left(-\frac{1}{2Q}\left(y - \frac{fb}{x}\right)^2\right)$$

$$\overbrace{p(x) = \mathcal{N}(\widetilde{x}, \widetilde{S})}^{\text{prior for } x} = \frac{1}{\sqrt{2\pi\widetilde{S}}} \exp\left(-\frac{1}{2\widetilde{S}}(x - \widetilde{x})^2\right)$$

Depth estimate or more generally the 3D reconstruction from 2D images is the central issue of computer vision; which is somewhat from computer graphics tasks.

Posterior, EKF, . . .



MAP estimate $\widehat{x}_{\mathrm{map}} = \arg\min_x J(x)$

$$J(x) = \frac{1}{2Q}\left(y - \frac{fb}{x}\right)^2 + \frac{1}{2\widetilde{S}}(x - \widetilde{x})^2$$

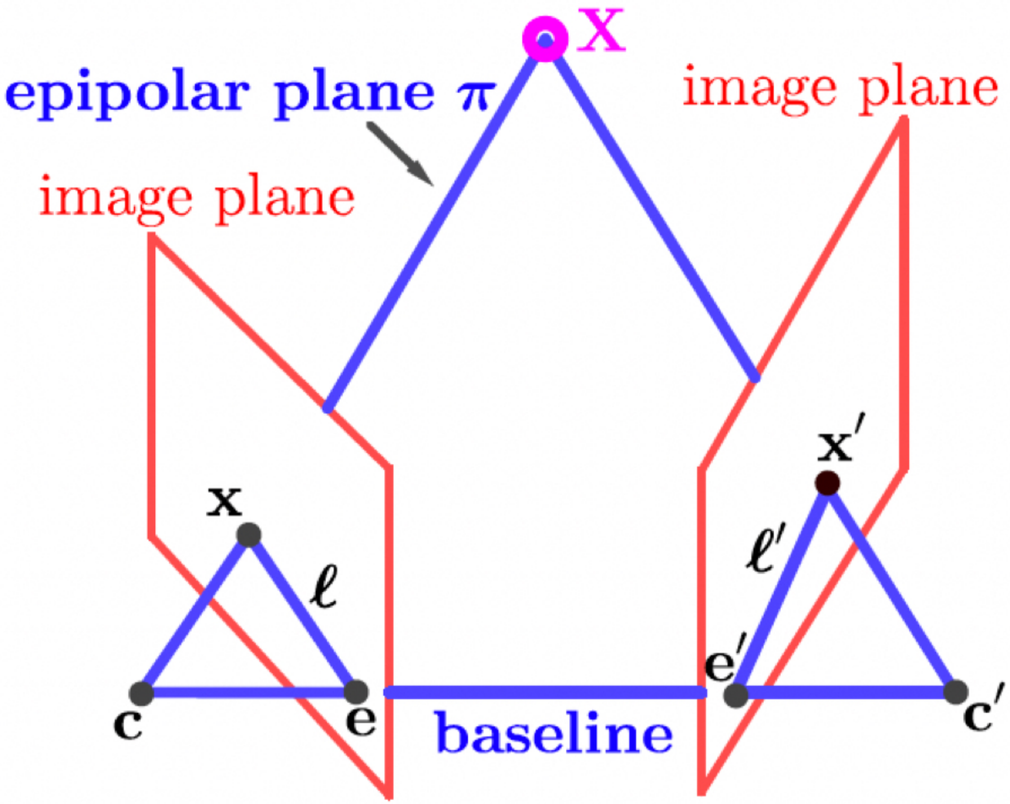
posterior mean $\bar{x} = \int dx\, x\, p(x|y) / \int dx\, p(x|y)$

Ex.: What is the ML solution?

$$\mathrm{E}[\delta x] = \frac{Qx^3}{f^2 b^2}$$

$$\rightarrow \widehat{x}_{\mathrm{corr}} = \widehat{x} - \mathrm{E}[\delta\widehat{x}] = \widehat{x} - \frac{Q\widehat{x}^3}{f^2 b^2}$$

Epipolar geometry



X: 3D point

$\mathbf{x} = \mathbf{P}\mathbf{X},\ \mathbf{P} = \mathbf{K}[\mathbf{R}|\mathbf{t}]$

$\mathbf{X} = \mathbf{P}^{+}\mathbf{x}$

$\vec{\ell} = \mathbf{x} \times \mathbf{x}'$

$$\vec{\ell'} = \overbrace{(\mathbf{P}'\mathbf{c})}^{\mathbf{e}'} \times (\mathbf{P}'\mathbf{P}^{+}\mathbf{x}) = (\mathbf{P}'\mathbf{c})^{\wedge}(\mathbf{P}'\mathbf{P}^{+}\mathbf{x})$$

define $\vec{\ell'} \equiv \mathbf{F}\mathbf{x}\,(\mathbf{x} \leftrightarrow \vec{\ell'}) \rightarrow \mathbf{F} = \mathbf{e}'^{\wedge}\mathbf{P}'\mathbf{P}^{+}$

if $\mathbf{x}$ lies on $\vec{\ell} \rightarrow \mathbf{x}^{\top}\vec{\ell} = 0$

$\rightarrow \mathbf{x}'$ lies on $\vec{\ell'} = \mathbf{F}\mathbf{x} \rightarrow 0 = \mathbf{x}'^{\top}\vec{\ell'}$

$\rightarrow \langle \mathbf{x}'|\mathbf{F}|\mathbf{x}\rangle = \mathbf{x}'^{\top}\mathbf{F}\mathbf{x} = 0$

F=fundamental matrix: $\{\mathbf{x}^{(i)} \leftrightarrow \mathbf{x}'^{(i)}\}$

How to calculate the fundamental matrix?

$$\mathbf{x}^{(i)} \leftrightarrow \mathbf{x}'^{(i)} : \qquad \mathbf{x} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}, \mathbf{x}' = \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}$$

$$xx'f_1 + xy'f_2 + xf_3 + yx'f_4 + yy'f_5 + yf_6 + x'f_7 + y'f_8 + f_9 = 0,\ \mathbf{f} = \begin{pmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \\ f_7 & f_8 & f_9 \end{pmatrix}$$

$$\begin{pmatrix} x/\xi, & y/\xi, & 1 \end{pmatrix} \begin{pmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \\ f_7 & f_8 & f_9 \end{pmatrix} \begin{pmatrix} x'/\xi \\ y'/\xi \\ 1 \end{pmatrix} = 0 \rightarrow \langle \vec{\zeta} | \mathbf{f} \rangle = 0$$

$$\vec{\zeta}^\top = \left(xx', xy', \xi x, yx', yy', \xi y, \xi x', \xi y', \xi^2\right)^\top, \ \mathbf{f}^\top = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9)$$

7-point algorithm, algebraic distance

$$(x^{(i)}x'^{(i)}, x^{(i)}y'^{(i)}, \xi x^{(i)}, y^{(i)}x'^{(i)}, y^{(i)}y'^{(i)}, \xi y^{(i)}, \xi x'^{(i)}, \xi y'^{(i)}, \xi^2)\,\mathbf{f} = 0,\ i = 1 \sim m$$

$$\mathbf{A}\mathbf{f} = \vec{0} : \mathbf{A} = \begin{pmatrix} x^{(1)}x'^{(1)} & x^{(1)}y'^{(1)} & \xi x^{(1)} & y^{(1)}x'^{(1)} & y^{(1)}y'^{(1)} & \xi y^{(1)} & \xi x'^{(1)} & \xi y'^{(1)} & \xi^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x^{(m)}x'^{(m)} & x^{(m)}y'^{(m)} & \xi x^{(m)} & y^{(m)}x'^{(m)} & y^{(m)}y'^{(m)} & \xi y^{(m)} & \xi x'^{(m)} & \xi y'^{(m)} & \xi^2 \end{pmatrix}$$

$$\mathbf{A} \in \mathrm{R}^{m\times 9}$$

comments:

(1) $\mathbf{F}$ is homogeneous: $\|\mathbf{F}\|_2 \equiv \left(\sum_{i=1}^{9} f_i^2\right)^{1/2} = 1$

(2) no exact solution because $\mathrm{rank}(\mathbf{A}) = 9$

(3) epipolar constraint $\rightarrow \det \mathbf{F} = 0$

$$\mathbf{f}^* = \mathrm{argmin}_{\mathbf{f}} \|\mathbf{A}\mathbf{f}\|^2 \leftrightarrow \text{subject to } \mathbf{f}^\top \mathbf{f} = 1$$

$$\min_{\mathbf{f}} \|\mathbf{A}\mathbf{f}\|^2 \leftrightarrow \text{smallest eigenvalue } \lambda \text{ of } \mathbf{A}^\top\mathbf{A}$$

Ex.: Are they equivalent?

Some improvements

weighted LS:

$$\mathbf{M} = \frac{1}{m}\sum_{i=1}^{m} w^{(i)} \vec{\zeta}^{(i)} \vec{\zeta}^{(i),\top}$$

geometrical distance:

$$S^{(i)} \approx \frac{\alpha^2(\langle \mathbf{x}^{(i)}|\mathbf{F}|\mathbf{x}'^{(i)}\rangle)^2}{\|\mathbf{P_k}\mathbf{F}\mathbf{x}'^{(i)}\|^2 + \|\mathbf{P_k}\mathbf{F}^\top\mathbf{x}^{(i)}\|^2}$$

$$\mathbf{P_k} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

requirement of $\det \mathbf{F} = 0$:

∗$\mathbf{F}$ using algebraic/geometrical distance

∗SVD of $\mathbf{F}$ in the form:

$$\mathbf{F} = \mathbf{U}\mathrm{diag}(\sigma_1, \sigma_2, \sigma_3)\mathbf{V}^\top$$

$\sigma_1 \geq \sigma_2 \geq \sigma_3$, $\mathbf{U}$ and $\mathbf{V}$ orthogonal

∗correct $\mathbf{F}$ as:

$$\mathbf{F} \leftarrow \mathbf{U}\mathrm{diag}\left(\frac{\sigma_1}{\sqrt{a^2}}, \frac{\sigma_2}{\sqrt{a^2}}, 0\right)\mathbf{V}^\top$$

$$a^2 = \sigma_1^2 + \sigma_2^2$$

Quiz 3: 5/6/2026

Quiz 3.1:

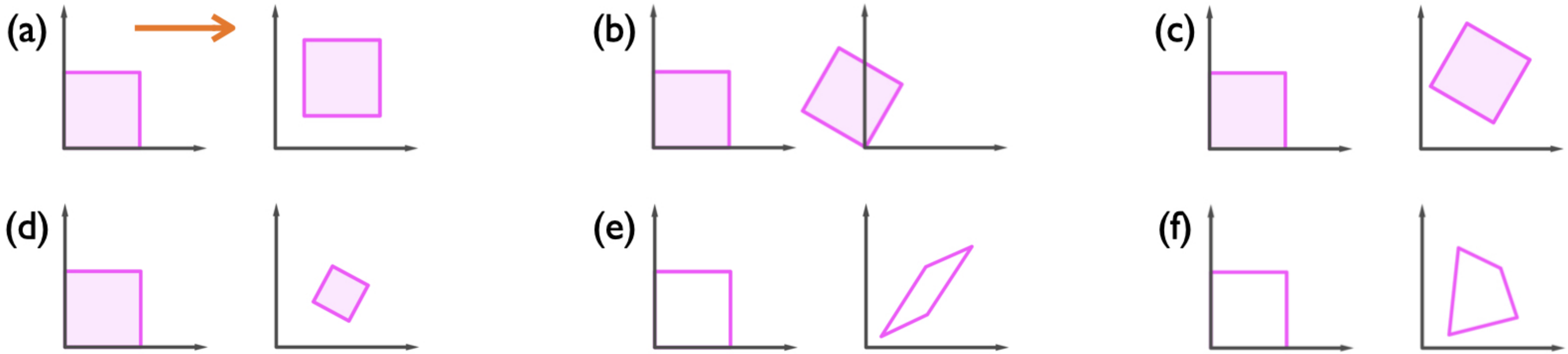
Explain the relationship between MAP/ML estimate and the LS, write down relevant expressions.

Quiz 3.2:

For 1D random walk, $x_k = x_{k-1} + w_k$ and $y_k = x_k + n_k$, write down the Kalman filter equations.

Quiz 3.3:

If $\mathbf{x}' = \mathbf{H}\mathbf{x}$, write down the expressions for $\mathbf{H}$, here $\mathbf{x}' = (x', y', 1)^\top$ and $\mathbf{x} = (x, y, 1)^\top$ are images.



Triangulation and re-projection error

$$\hat{\mathbf{x}}_1^*, \hat{\mathbf{x}}_2^* = \mathrm{argmin}_{\hat{x}_1,\hat{x}_2} \overbrace{s^2(\mathbf{x}_1,\mathbf{x}_2)}^{\text{some distance}}, \text{ subject to } \langle \hat{\mathbf{x}}_1 | \mathbf{F} | \hat{\mathbf{x}}_2 \rangle = 0$$

fundamental matrix $\mathbf{F}$:

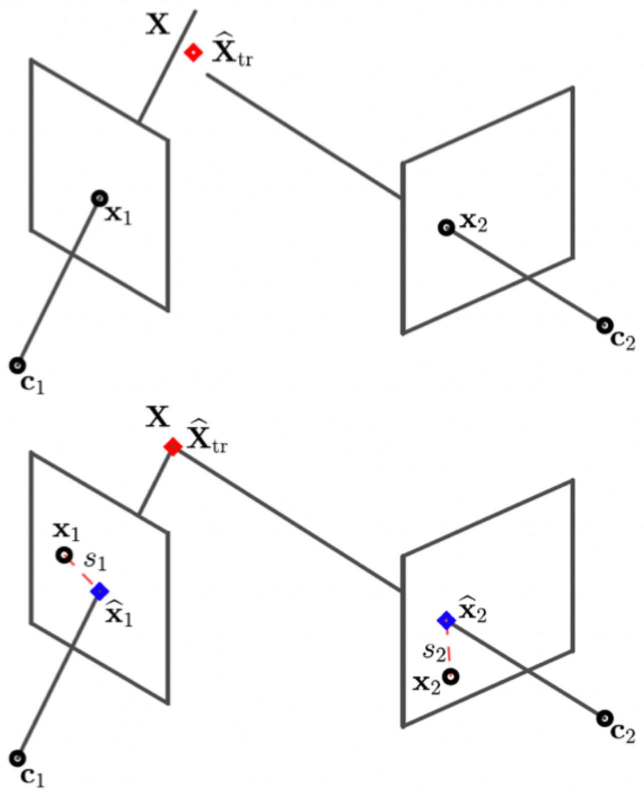
∗unit vector along $\mathbf{c}_j\mathbf{X}$: $\mathbf{x}_j^0 = \dfrac{\mathbf{R}_j^{-1}\mathbf{K}_j^{-1}\mathbf{x}_j}{\|\mathbf{R}_j^{-1}\mathbf{K}_j^{-1}\mathbf{x}_j\|}, \; j = 1,2$

∗estimate for $\mathbf{X}$: $\mathbf{q}_j = \mathbf{c}_j + d_j\mathbf{x}_j^0$

∗error function $e(\mathbf{X}) = \sum_{j=1,2} r_j^2(\mathbf{X}), \; r_j^2(\mathbf{X}) = \|\mathbf{q}_j - \mathbf{X}\|^2$

$$\hat{\mathbf{X}}_{tr} = \left[\sum_{j=1,2}\left(\vec{1} - \mathbf{x}_j^0\mathbf{x}_j^{0,\top}\right)\right]^{-1}\left[\sum_{j=1,2}\left(\vec{1} - \mathbf{x}_j^0\mathbf{x}_j^{0,\top}\right)\mathbf{c}_j\right]$$

$$\text{error(images)} = \sum_{j=1,2} |\mathbf{x}_j - \hat{\mathbf{x}}_j|^2$$



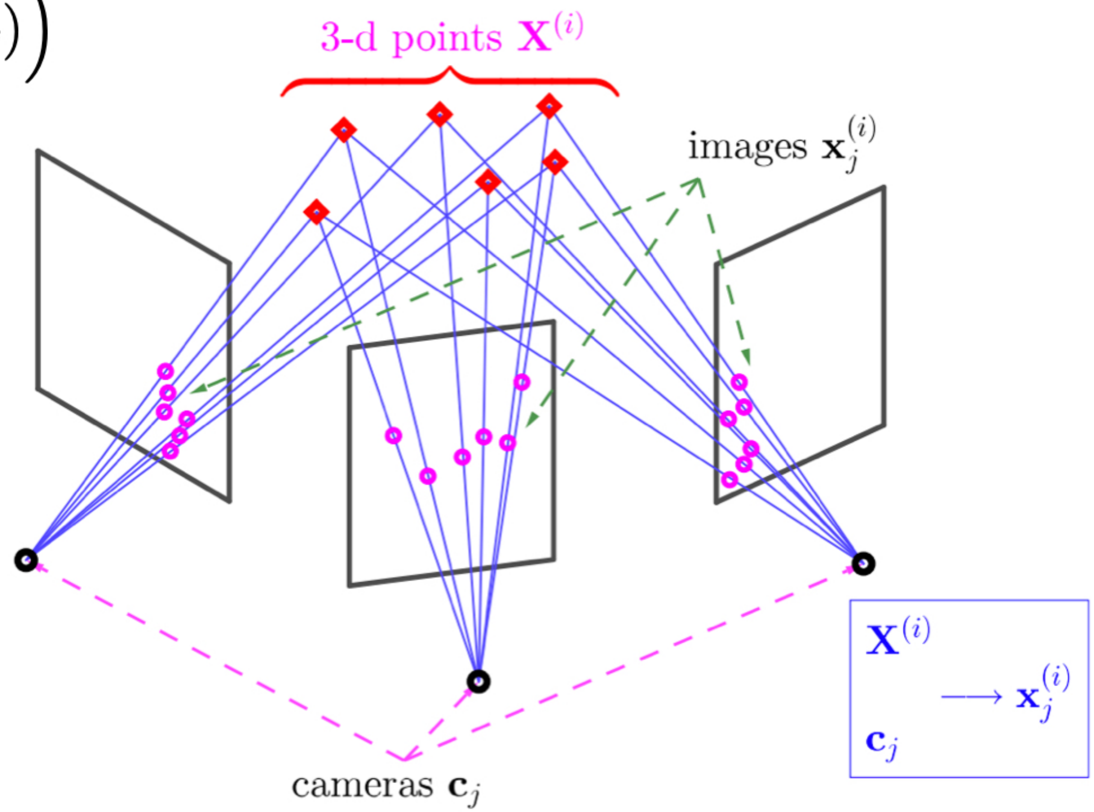
More cameras and more 3D points

$$J(\vec{\Phi}) = \frac{1}{2} \sum_{i=1\sim m, j=1\sim n} \left(\mathbf{x}_j^{(i)} - \widehat{\mathbf{x}}_j^{(i)}(\mathbf{X}^{(i)}, \vec{\Theta}_j) \right)^2$$

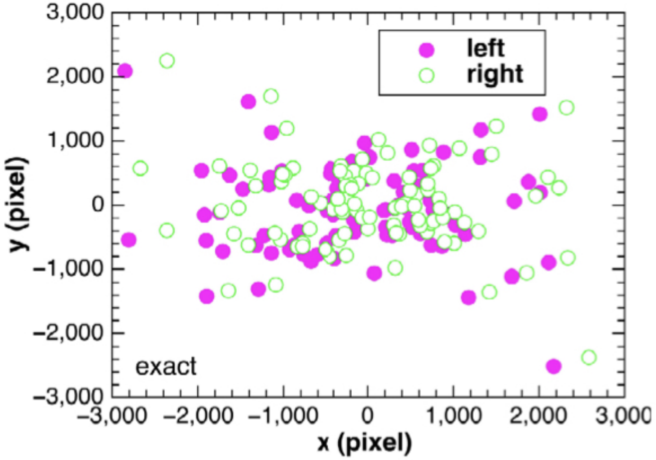
$$\vec{\Phi} = \begin{pmatrix} \vec{\Phi}_{\mathrm{p}} \\ \vec{\Phi}_{\mathrm{cam}} \end{pmatrix} \in \mathrm{R}^{3m+6n}, \ \vec{\Phi}_{\mathrm{p}} = \begin{pmatrix} \mathbf{X}^{(1)} \\ \mathbf{X}^{(2)} \\ \cdots \\ \mathbf{X}^{(m)} \end{pmatrix} \in \mathrm{R}^{3m}$$

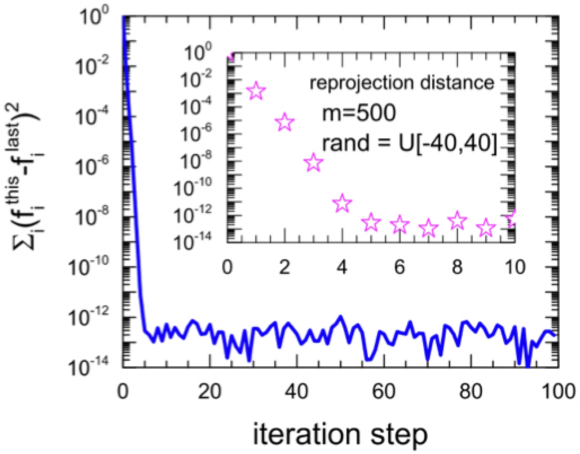
$$\vec{\Phi}_{\mathrm{cam}} = \begin{pmatrix} \vec{\Theta}_1 \\ \vec{\Theta}_2 \\ \cdots \\ \vec{\Theta}_n \end{pmatrix} \in \mathrm{R}^{6n}, \ \vec{\Theta}_j = (\mathbf{R}_j, \mathbf{t}_j) \in \mathrm{R}^6$$

- 3D reconstruction
- structure from motion
- SLAM

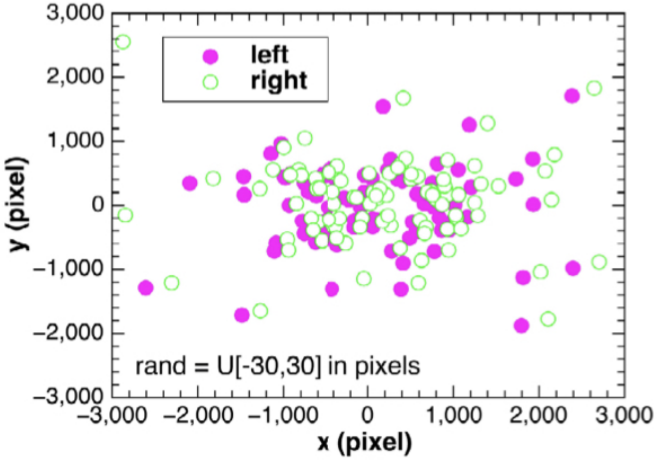


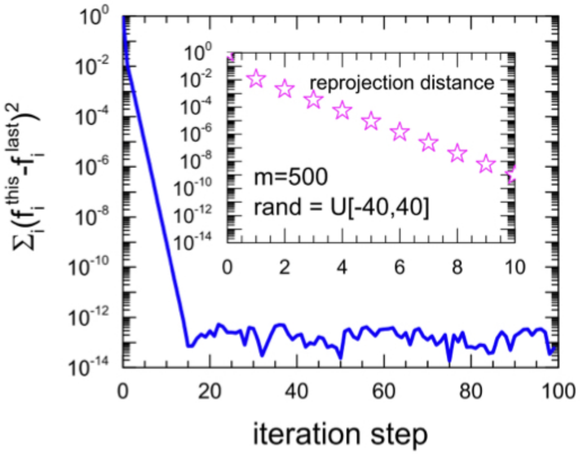
Numerical example





$$e(\text{iteration step}) = \sum_{i=1}^{9} \left(f_i^{\text{this}} - f_i^{\text{last}}\right)^2,$$



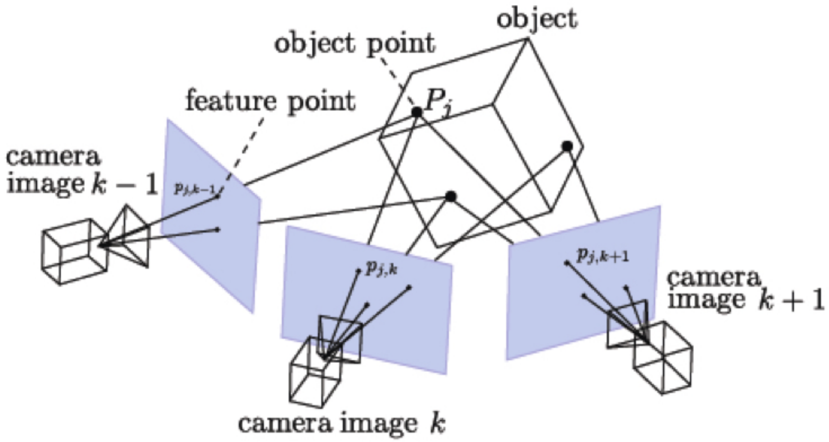


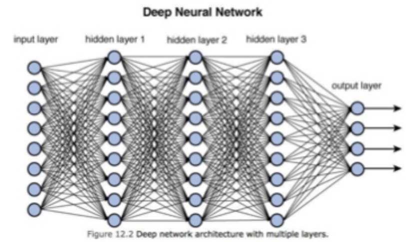
$$\mathbf{F} = \begin{pmatrix} 0 & 0 & -1/\sqrt{20} \\ 0 & 0 & 3/\sqrt{20} \\ 1/\sqrt{20} & -3/\sqrt{20} & 0 \end{pmatrix}$$

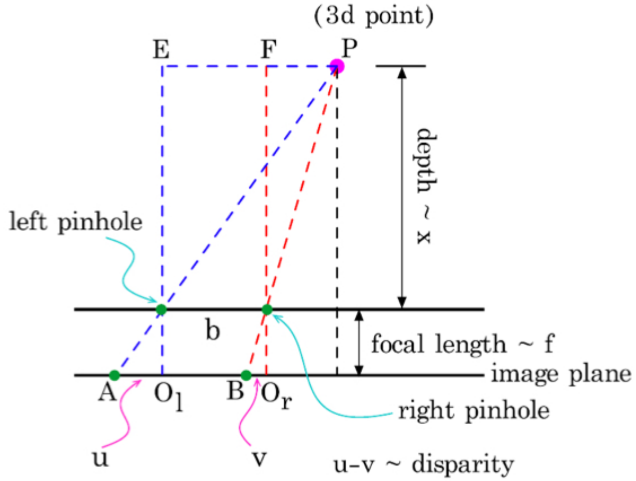
$$\mathbf{F}^{\text{LS}} = \begin{pmatrix} -0.000358 & -0.009160 & -0.241458 \\ 0.009391 & 0.002762 & 0.666575 \\ 0.241056 & -0.662619 & 0.000489 \end{pmatrix}$$

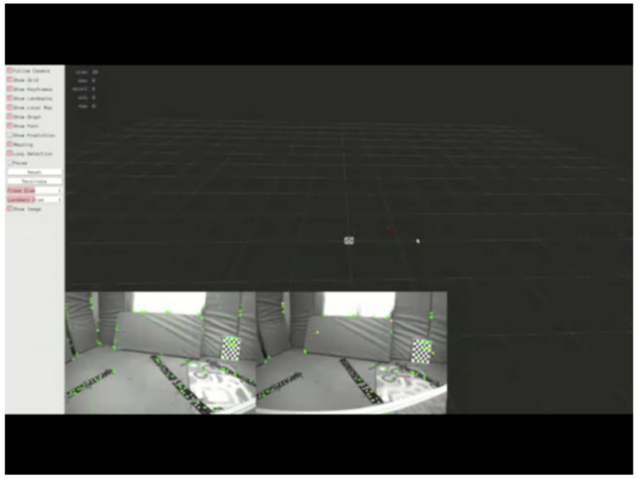
$$\mathbf{F}^{\text{RP}} = \begin{pmatrix} -0.000335 & -0.013974 & -0.243422 \\ 0.014273 & 0.002731 & 0.667123 \\ 0.242080 & -0.660780 & 0.005227 \end{pmatrix}$$

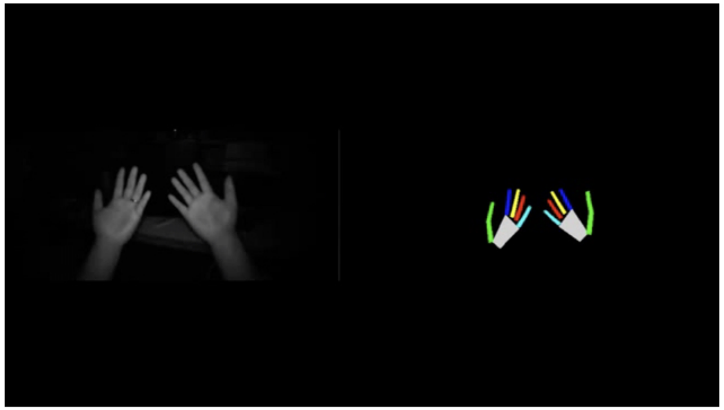
Localization and tracking



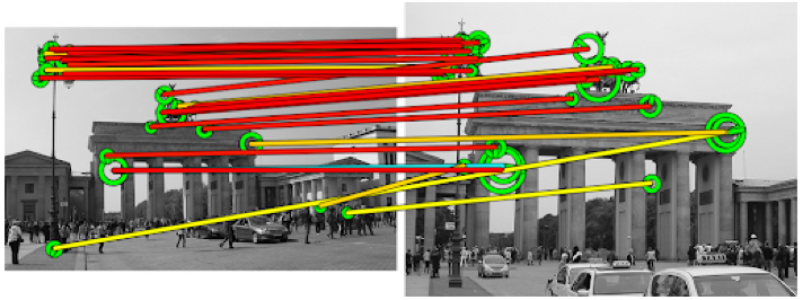


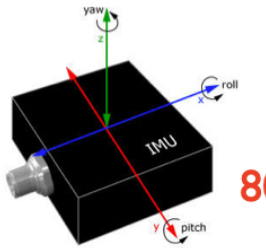




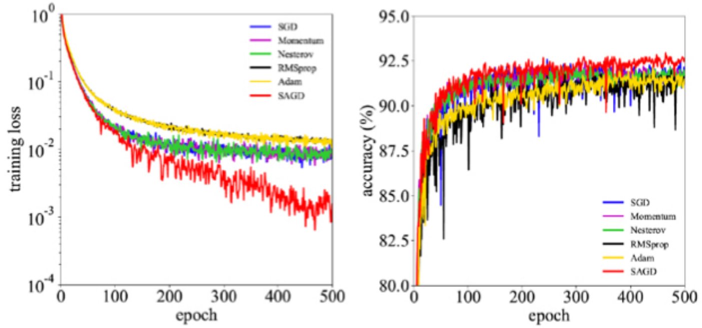


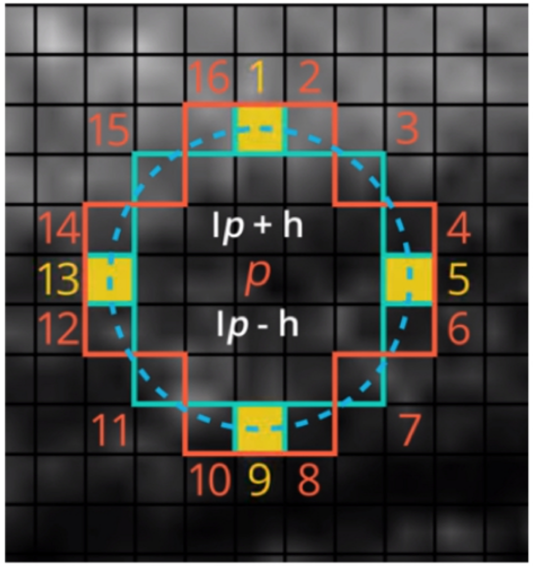
Reconstruction and data fusion





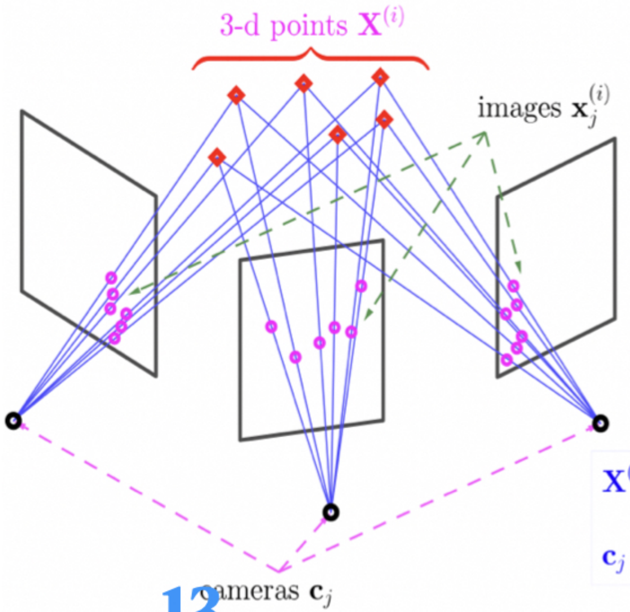
800-1000Hz

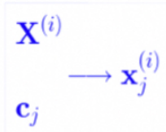




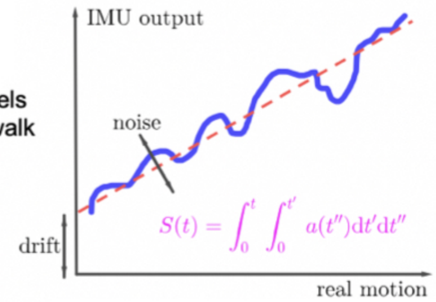
~30-90fps

SLAM, RECONSTURCTION





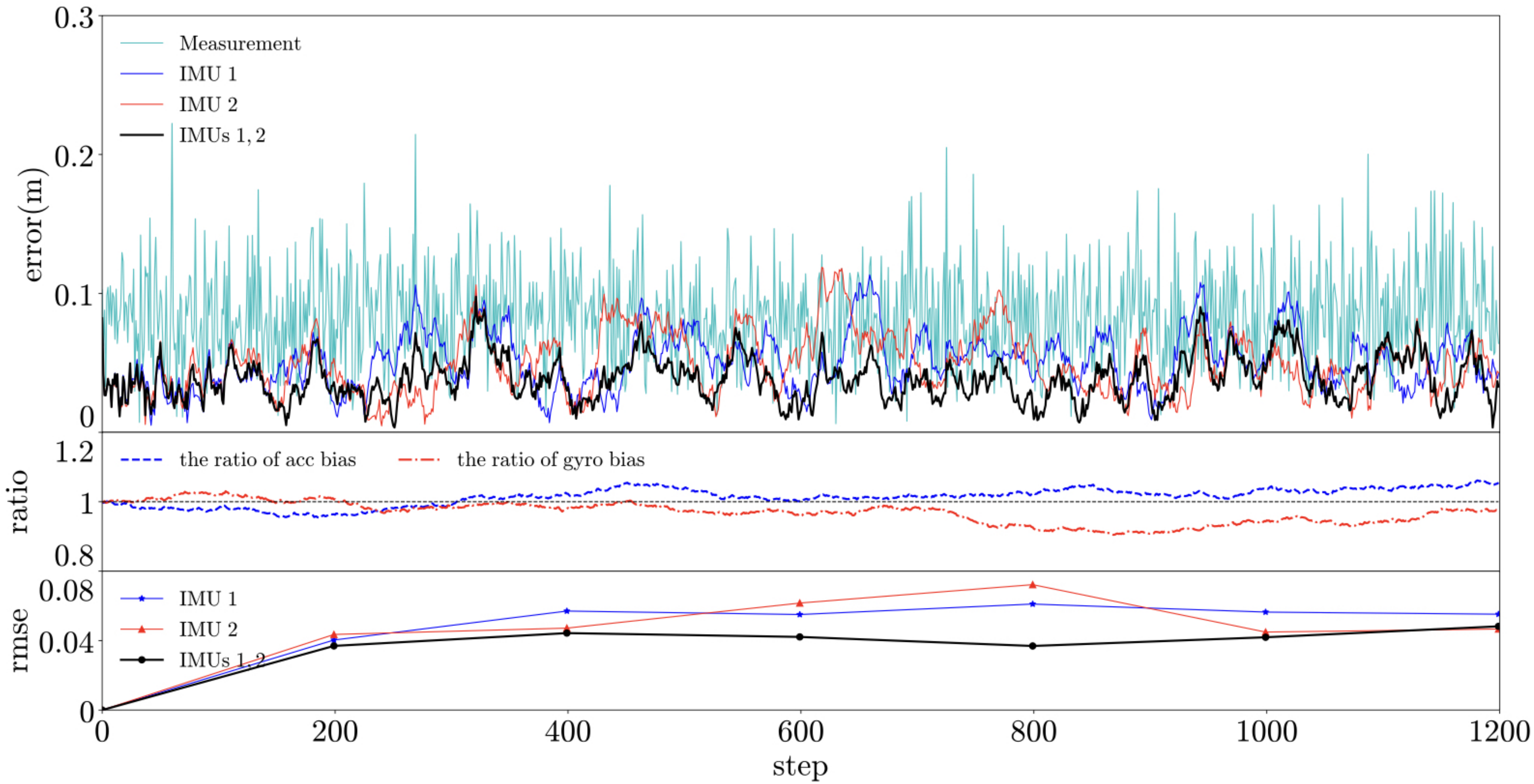
bias models
random walk



fast/accurate algorithm

- uncertain $\mathbf{c}_j$ and $\mathbf{x}_j^{(i)}$
  3d reconstruction
- uncertain $\mathbf{X}^{(i)}$ and $\mathbf{x}_j^{(i)}$
  motion determination
- parts of $\mathbf{X}^{(i)}/\mathbf{x}_j^{(i)}$ known
  SLAM

Example: multiple IMU optimization



*Solution of normal equation

*normal equation $\mathbf{J}^\top \mathbf{J}\vec{\delta} = \mathbf{J}^\top \vec{\epsilon} \rightarrow \mathbf{J}^\top \vec{\Sigma}_{\mathbf{M}}^{-1} \mathbf{J}\vec{\delta} = \mathbf{J}^\top \vec{\Sigma}_{\mathbf{M}}^{-1} \vec{\epsilon}$ (with measurement uncertainty)

*prediction for measurement $\widehat{\mathbf{M}} = \mathbf{f}(\mathbf{P})$, $\mathbf{P} = \left(\mathbf{a}^\top, \mathbf{b}^\top\right)^\top$, Jacobian $\mathbf{J} = (\mathbf{A}, \mathbf{B}) = \left(\frac{\partial \widehat{\mathbf{M}}}{\partial \mathbf{a}}, \frac{\partial \widehat{\mathbf{M}}}{\partial \mathbf{b}}\right)$

$$\begin{pmatrix} \mathbf{A}^\top \vec{\Sigma}_{\mathbf{M}}^{-1} \mathbf{A} & \mathbf{A}^\top \vec{\Sigma}_{\mathbf{M}}^{-1} \mathbf{B} \\ \mathbf{B}^\top \vec{\Sigma}_{\mathbf{M}}^{-1} \mathbf{A} & \mathbf{B}^\top \vec{\Sigma}_{\mathbf{M}}^{-1} \mathbf{B} \end{pmatrix} \begin{pmatrix} \vec{\delta}_{\mathbf{a}} \\ \vec{\delta}_{\mathbf{b}} \end{pmatrix} = \begin{pmatrix} \mathbf{A}^\top \vec{\Sigma}_{\mathbf{M}}^{-1} \vec{\epsilon} \\ \mathbf{B}^\top \vec{\Sigma}_{\mathbf{M}}^{-1} \vec{\epsilon} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{U} & \mathbf{W} \\ \mathbf{W}^\top & \mathbf{V} \end{pmatrix} \begin{pmatrix} \vec{\delta}_{\mathbf{a}} \\ \vec{\delta}_{\mathbf{b}} \end{pmatrix} = \begin{pmatrix} \vec{\epsilon}_{\mathbf{a}} \\ \vec{\epsilon}_{\mathbf{b}} \end{pmatrix}, \ \mathbf{U} = \mathbf{A}^\top \vec{\Sigma}_{\mathbf{M}}^{-1} \mathbf{A},\ \mathbf{W} = \mathbf{A}^\top \vec{\Sigma}_{\mathbf{M}}^{-1} \mathbf{B},\ \mathbf{V} = \mathbf{B}^\top \vec{\Sigma}_{\mathbf{M}}^{-1} \mathbf{B}$$

$$\left(\mathbf{U} - \mathbf{W}\mathbf{V}^{-1}\mathbf{W}^\top\right) \vec{\delta}_{\mathbf{a}} = \vec{\epsilon}_{\mathbf{a}} - \mathbf{W}\mathbf{V}^{-1} \vec{\epsilon}_{\mathbf{b}},\ \mathbf{V}\vec{\delta}_{\mathbf{b}} = \vec{\epsilon}_{\mathbf{b}} - \mathbf{W}^\top \vec{\delta}_{\mathbf{a}}$$

*Bundle adjustment (BA)

$\mathbf{x}_j^{(i)} \leftrightarrow \mathbf{M}$, camera $\mathbf{a} = (\mathbf{a}_1^\top, \mathbf{a}_2^\top, \cdots, \mathbf{a}_n^\top)^\top$, $\mathbf{b}^{(i)} =$ *i*th 3D point

$$\mathbf{A}^{(i)} = \operatorname{diag}\left(\mathbf{A}_1^{(i)}, \mathbf{A}_2^{(i)}, \cdots, \mathbf{A}_n^{(i)}\right), \ \mathbf{B}^{(i)} = \left(\mathbf{B}_1^{(i),\top}, \mathbf{B}_2^{(i),\top}, \cdots, \mathbf{B}_n^{(i),\top}\right)^\top$$

$$\mathbf{A}_j^{(i)} = \partial \hat{\mathbf{x}}_j^{(i)} / \partial \mathbf{a}_j, \ \mathbf{B}_j^{(i)} = \partial \hat{\mathbf{x}}_j^{(i)} / \partial \mathbf{b}^{(i)}$$

$$\boxed{\mathbf{U}_j = \sum_i \mathbf{A}_j^{(i),\top} \vec{\Sigma}_{\mathbf{x}_j^{(i)}}^{-1} \mathbf{A}_j^{(i)}, \ \mathbf{V}^{(i)} = \sum_j \mathbf{B}_j^{(i),\top} \vec{\Sigma}_{\mathbf{x}_j^{(i)}}^{-1} \mathbf{B}_j^{(i)}, \ \mathbf{W}_j^{(i)} = \mathbf{A}_j^{(i),\top} \vec{\Sigma}_{\mathbf{x}_j^{(i)}}^{-1} \mathbf{B}_j^{(i)}, \ \mathbf{Y}_j^{(i)} = \mathbf{W}_j^{(i)} \mathbf{V}^{(i),-1}}$$

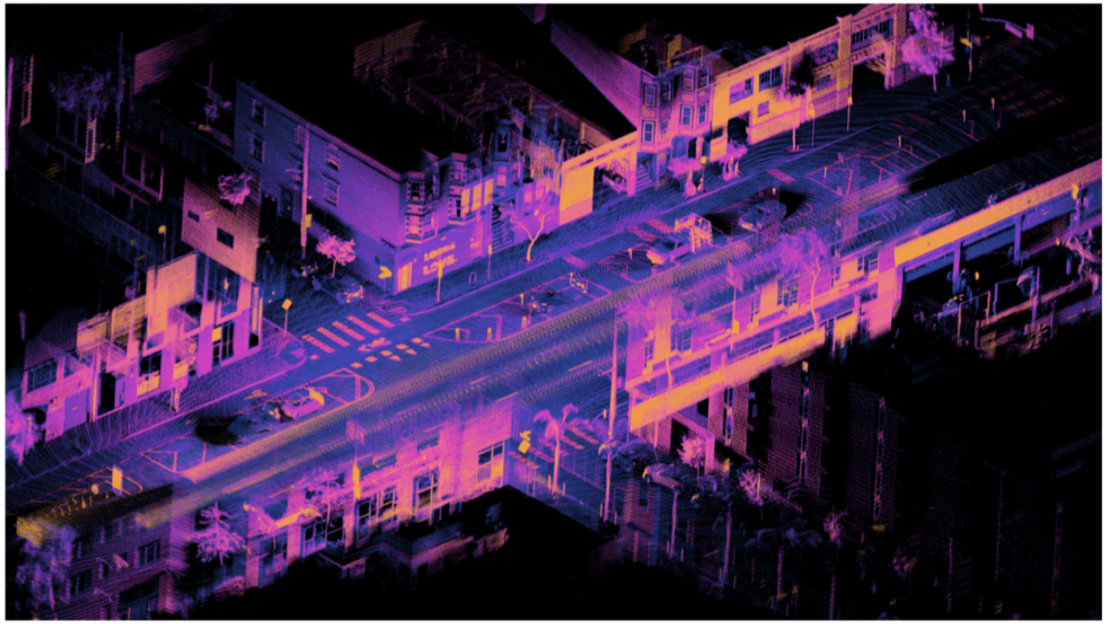
$$\boxed{\vec{\epsilon}_{\mathbf{a}_j} = \sum_i \mathbf{A}_j^{(i),\top} \vec{\Sigma}_{\mathbf{x}_j^{(i)}}^{-1} \vec{\epsilon}_j^{(i)}, \ \vec{\epsilon}_{\mathbf{b}^{(i)}} = \sum_j \mathbf{B}_j^{(i),\top} \vec{\Sigma}_{\mathbf{x}_j^{(i)}}^{-1} \vec{\epsilon}_j^{(i)}}$$

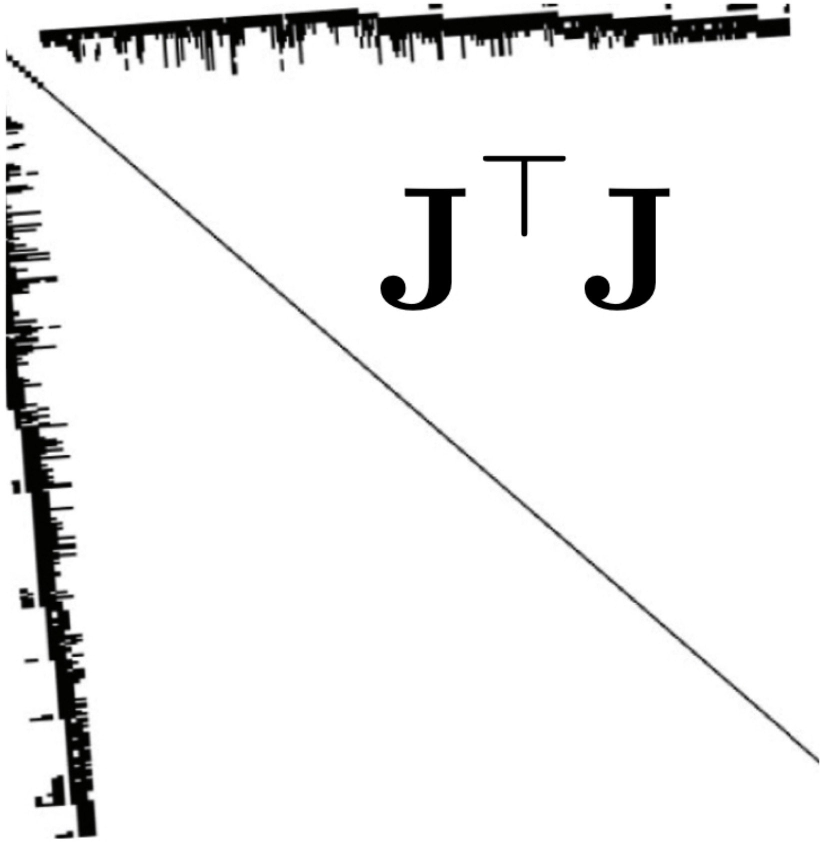
$$\boxed{\mathbf{S}\vec{\delta}_{\mathbf{a}} = (\mathbf{e}_1^\top, \cdots, \mathbf{e}_n^\top)^\top, \ \mathbf{S}_{jj} = -\sum_i \mathbf{Y}_j^{(i)} \mathbf{W}_j^{(i),\top} + \mathbf{U}_j, \ \mathbf{S}_{jk} = -\sum_i \mathbf{Y}_j^{(i)} \mathbf{W}_k^{(i),\top}, \ \mathbf{e}_j = \vec{\epsilon}_{\mathbf{a}_j} - \sum_i \mathbf{Y}_j^{(i)} \vec{\epsilon}_{\mathbf{b}^{(i)}}}$$

$$\boxed{\vec{\delta}_{\mathbf{b}^{(i)}} = \mathbf{V}^{(i),-1} \left(\vec{\epsilon}_{\mathbf{b}^{(i)}} - \sum_j \mathbf{W}_j^{(i),\top} \vec{\delta}_{\mathbf{a}_j}\right)}$$

Ex.: # of parameter?

SLAM and approximated Hessian





Ex.: Explain why?