Lecture 14

Fast Fourier Transform for Data Processing

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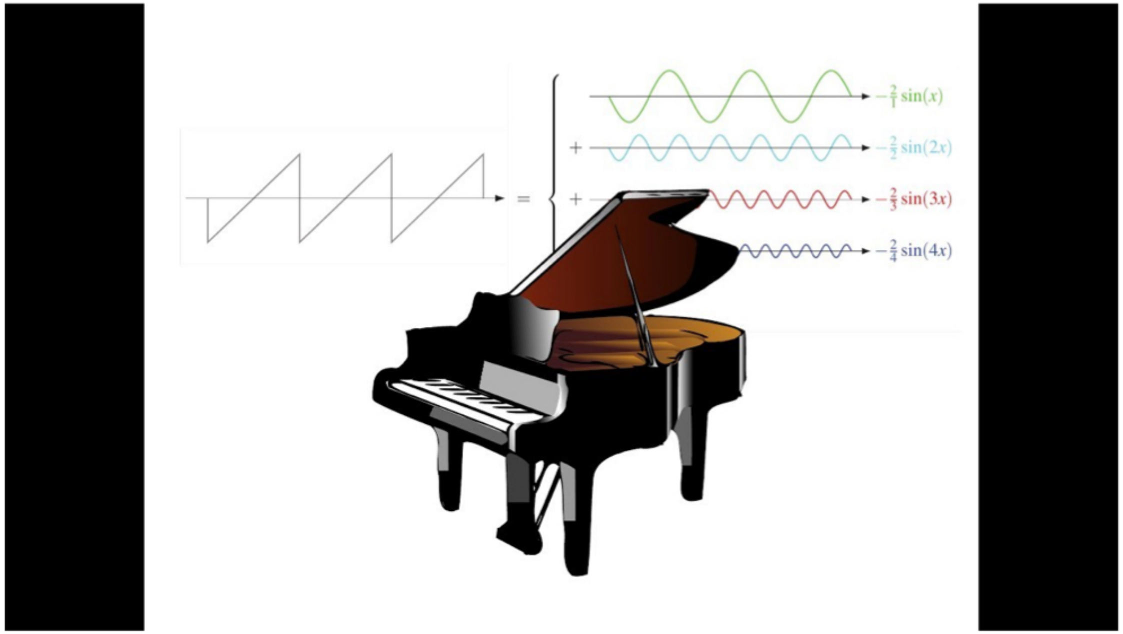
Introduction to Algorithms for Data Science and Physics IMP@Fudan, 2026

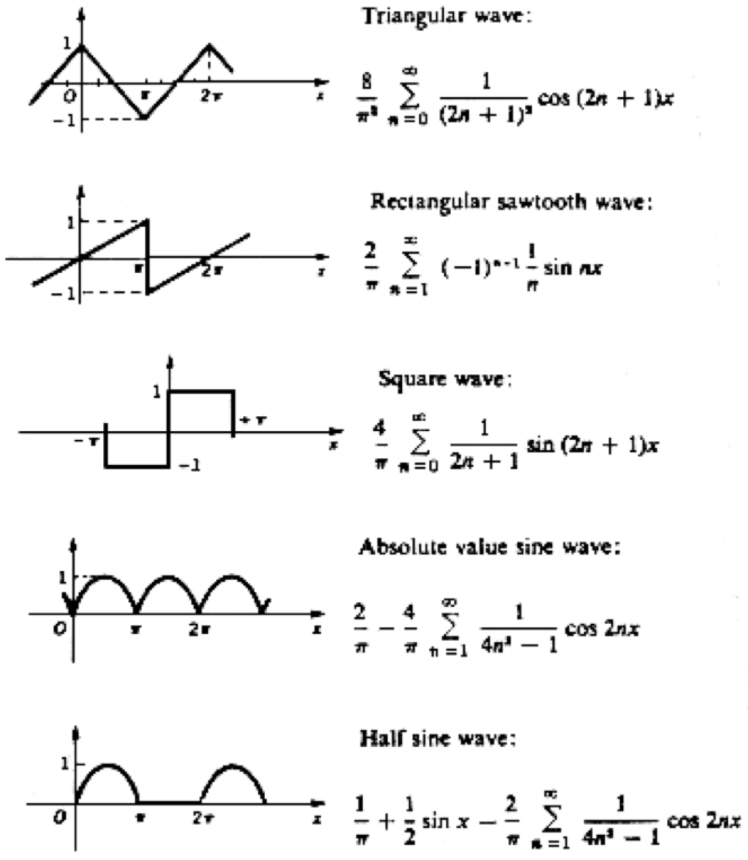
Topics of this lecture:

- Fourier series for approximating functions $\sum \text{harmonics} \leftrightarrow f(x)$
- Fourier transform using complex variables $f(x) = \sum_j c_j e^{ij\pi x/\ell}$
- solving partial differential equations $\partial^2 n(x,t)/\partial x^2 = D^{-1}\partial n(x,t)/\partial t$
- sampling, aliasing, Nyquist frequency, Shannon theorem $h_{\mathrm{WS}}(t) = \Delta \sum_{n=-\infty}^{\infty} h_n \frac{\sin[2\pi f_{\mathrm{c}}(t-n\Delta)]}{\pi(t-n\Delta)}$
- discrete Fourier transform (FFT) $H_k^{\mathrm{eoeeeo}\cdots\mathrm{oee}} = h_n$
- Fast Fourier transform using divide-and-conquer $\sim \mathcal{O}(N\log N)$

Motivation: review of Fourier series

$$\sum \text{harmonics} \leftrightarrow f(x)$$





Triangular wave:

$$\frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos(2n+1)x$$

Rectangular sawtooth wave:

$$\frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}\sin nx$$

Square wave:

$$\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin(2n+1)x$$

Absolute value sine wave:

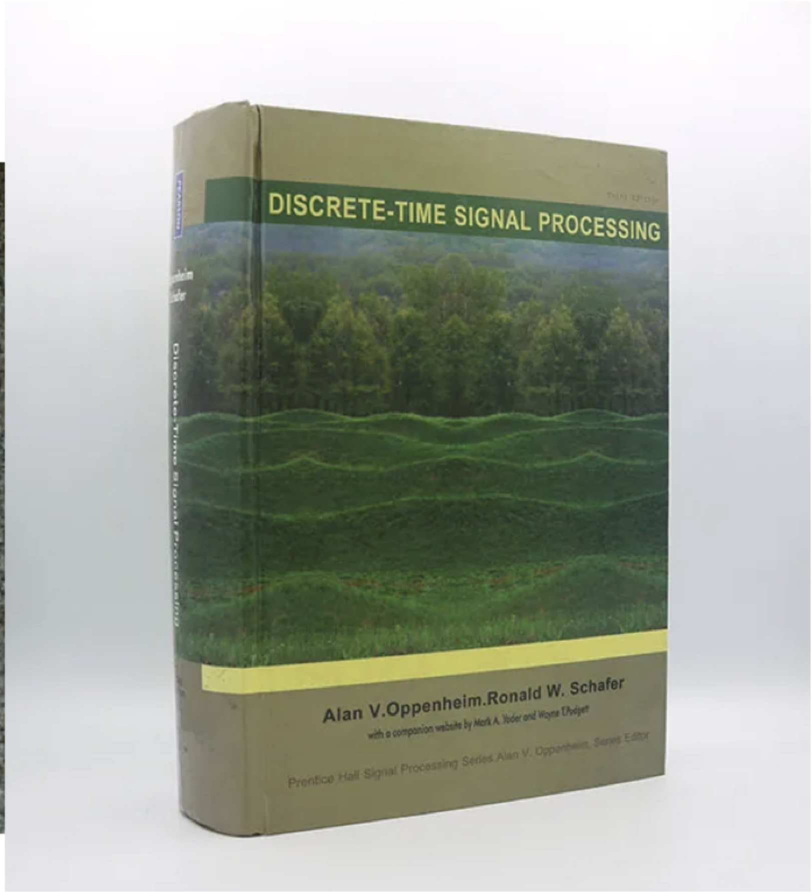
$$\frac{2}{\pi}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos 2nx$$

Half sine wave:

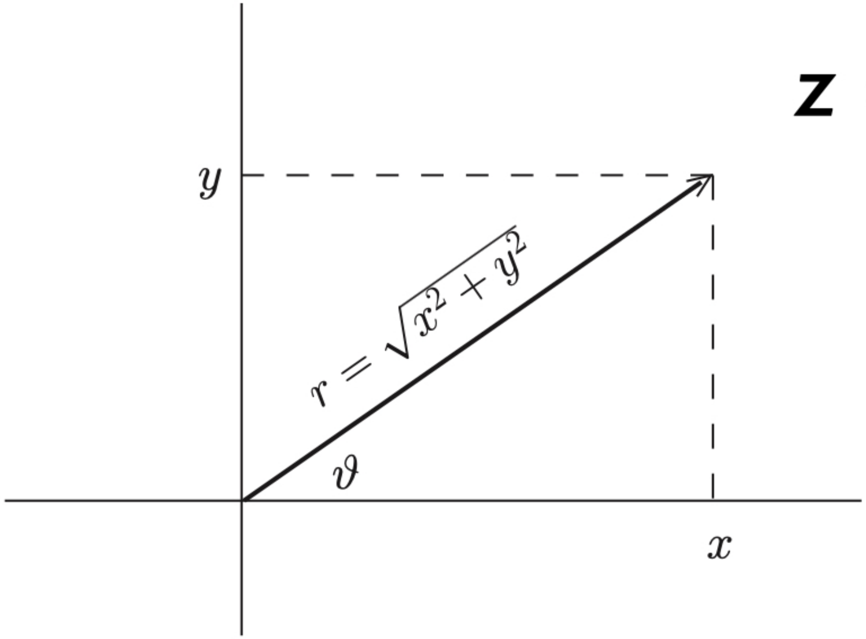
$$\frac{1}{\pi}+\frac{1}{2}\sin x-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos 2nx$$

signal and system problems

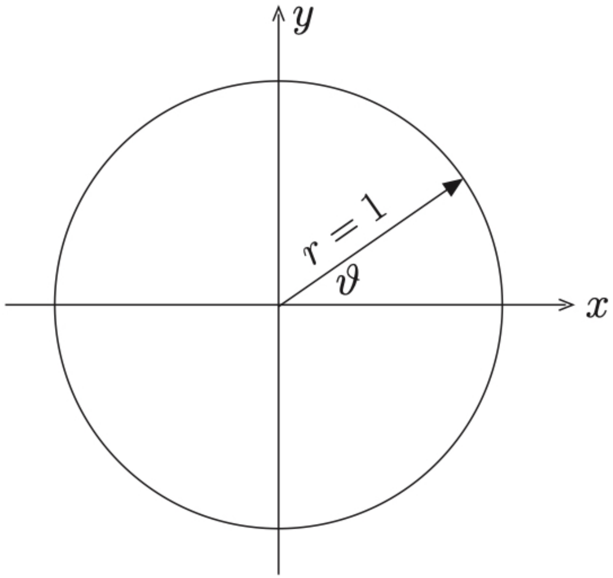
Widely used in signals and systems



Quick review of complex variables



$$z = re^{i\vartheta}$$



$$\lim_{K\to\infty} \int_{-K}^{K} \frac{1}{2\pi} e^{i\omega t} d\omega = \lim_{K\to\infty} \frac{1}{\pi} \frac{\sin Kx}{x}$$

Ex.: find the roots of $x^N = 1$ in complex domain.

Ex.: show $(n/\pi)[1 + n^2x^2]^{-1}$ is a δ function approximation.

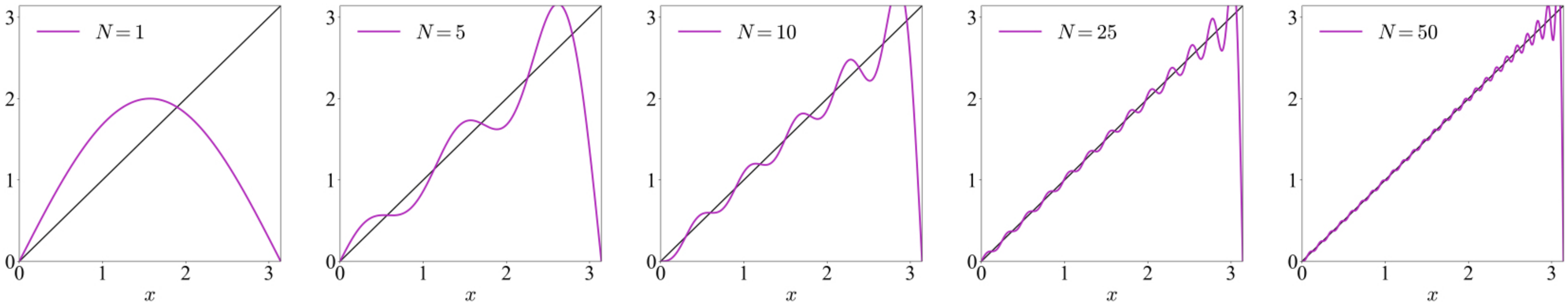
$$\delta(t) = \frac{1}{2\pi} \int e^{i\omega t} d\omega$$

Fourier series of continuous function

$$f(x) = \frac{a_0}{2} + \sum_{i=1}^{\infty} \left[a_i \cos\left(\frac{i\pi x}{\ell}\right) + b_i \sin\left(\frac{i\pi x}{\ell}\right) \right] \quad f(x+2\ell) = f(x)$$

$$a_i = \frac{1}{\ell}\int_{-\ell}^{\ell} f(\xi)\cos\left(\frac{i\pi\xi}{\ell}\right) d\xi, \ b_i = \frac{1}{\ell}\int_{-\ell}^{\ell} f(\xi)\sin\left(\frac{i\pi\xi}{\ell}\right) d\xi$$

example: $f(x) = x$



Ex.: what is the expansion?

Ex.: work out for $f(x) = \begin{cases} \sin \omega x, & 0 \leq x < T/2, \\ 0, & T/2 \leq x \leq T. \end{cases}$

Fourier transform using exponential

$$f(x) = \sum_{i=-\infty}^{\infty} f_i \exp\left(\frac{\sqrt{-1}i\pi x}{\ell}\right) = \sum_{i=-\infty}^{\infty} f_i \exp\left(2\pi\sqrt{-1}x \cdot \frac{i}{2\ell}\right)$$

$$\Big\downarrow \quad f \leftrightarrow H,\ x \leftrightarrow f,\ f_i \leftrightarrow h(t),\ i \leftrightarrow t$$

$$H(f) = \int h(t)e^{2\pi\sqrt{-1}ft}dt,\ \ h(t) = \int H(f)e^{-2\pi\sqrt{-1}ft}df$$

Ex.: write the FT pair $h(t)$ and $H(\omega)$.

t : time; $f = 1/T = 2\pi\omega$: frequency

FT: solving differential equations

diffusion equation: $$\frac{\partial^2 n(x,t)}{\partial x^2} = \frac{1}{D}\frac{\partial n(x,t)}{\partial t}$$

$$n(x,t) = \int \tilde{n}(k,t) e^{ikx} dk$$

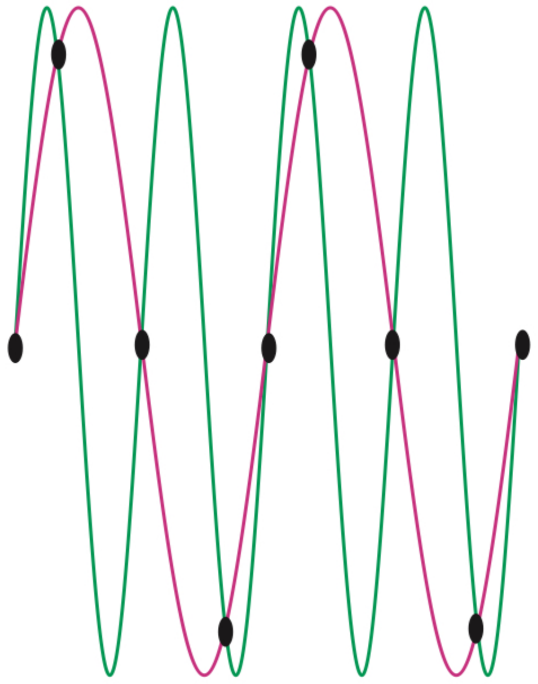
Ex.: if $n(x,0) = S$, work out $n(x,t)$.

$$\rightarrow (ik)^2 \tilde{n}(k,t) = \frac{1}{D}\frac{\partial \tilde{n}(k,t)}{\partial t} \rightarrow \tilde{n}(k,t) = \tilde{n}_0(k) e^{-Dk^2 t}$$

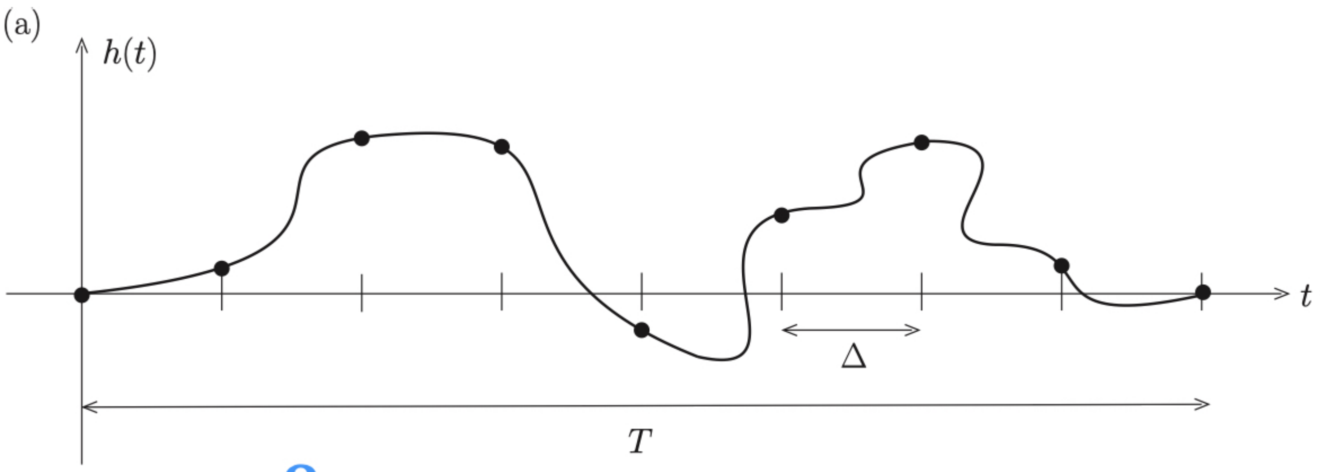
Ex.: solve the PDE $\partial^2\phi/\partial t^2 = c^2 \partial^2\phi/\partial x^2$ with $c = \sqrt{T/\rho}$.

Sampling rate, Nyquist critical frequency

$h_n = h(n\Delta),\ n = \cdots, -2, -1, 0, 1, 2, \cdots \rightarrow$ Nyquist frequency: $f_c = \dfrac{1}{2\Delta}$

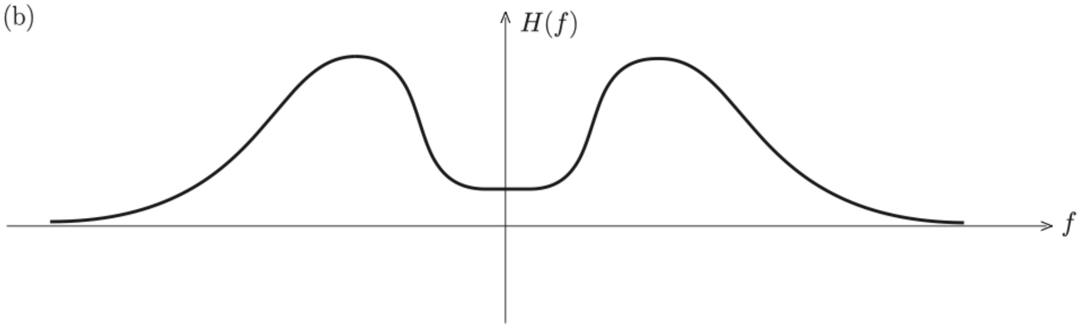


sampling rate $\Delta^{-1} > 2f_{\mathrm{max}}$

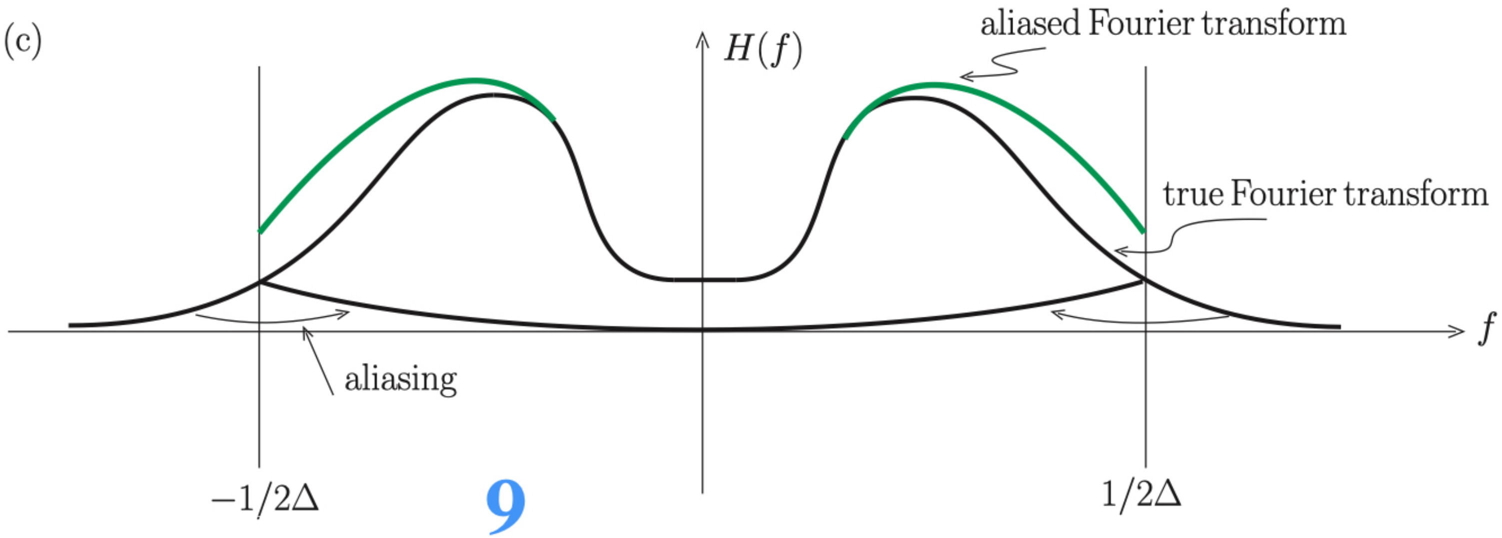


Aliasing

If we sample a continuous function that is not bandwidth limited to less than the Nyquist critical frequency, something will happen. In that case it turns out that all of the power spectral density that lies outside of the frequency -fc~fc is spuriously moved into that range. This phenomenon is called aliasing. Any frequency component outside of the frequency range -fc~fc is aliased, namely falsely translated into that range by the very act of discrete sampling.



$$\Delta^{-1} > 2f_{\max}$$

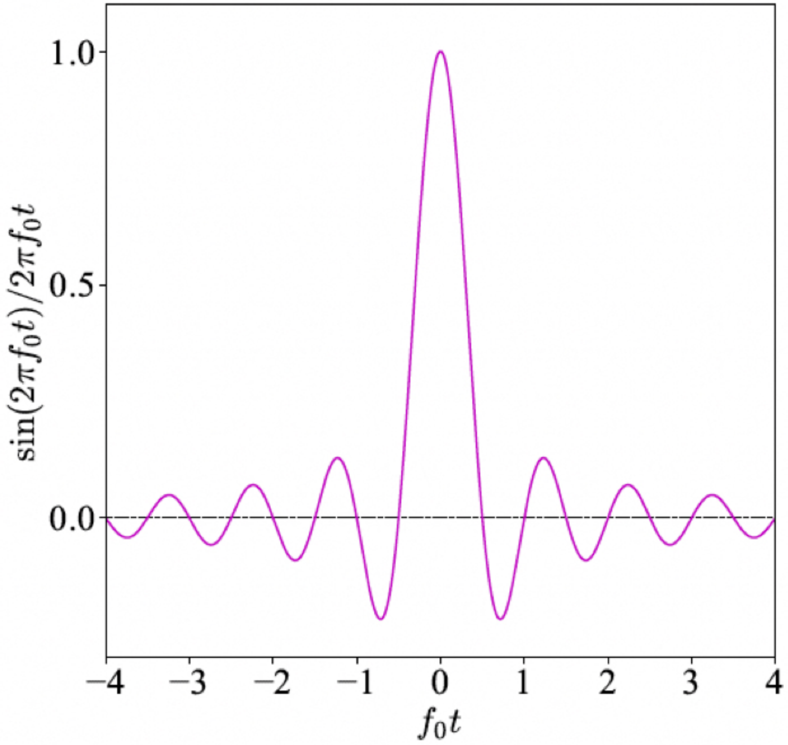


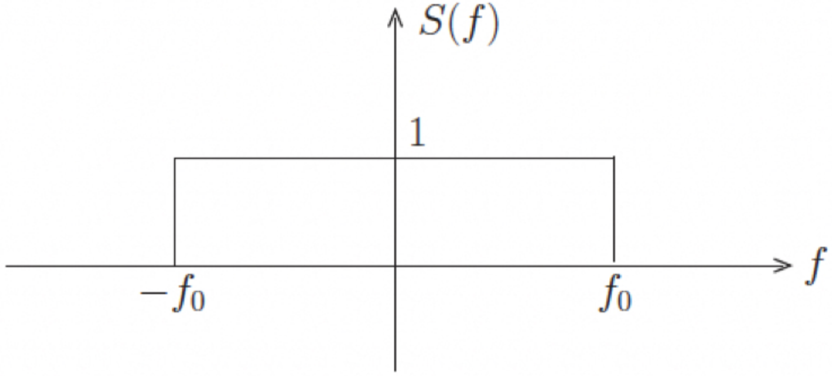
Whittaker-Shannon reconstruction

$$h_{\mathrm{WS}}(t) = \Delta \sum_{n=-\infty}^{\infty} h_n \frac{\sin[2\pi f_c(t-n\Delta)]}{\pi(t-n\Delta)}$$

$$= \Delta \sum_{n=-\infty}^{\infty} 2f_c h_n \frac{\sin[2\pi f_c(t-n\Delta)]}{2\pi f_c(t-n\Delta)}$$

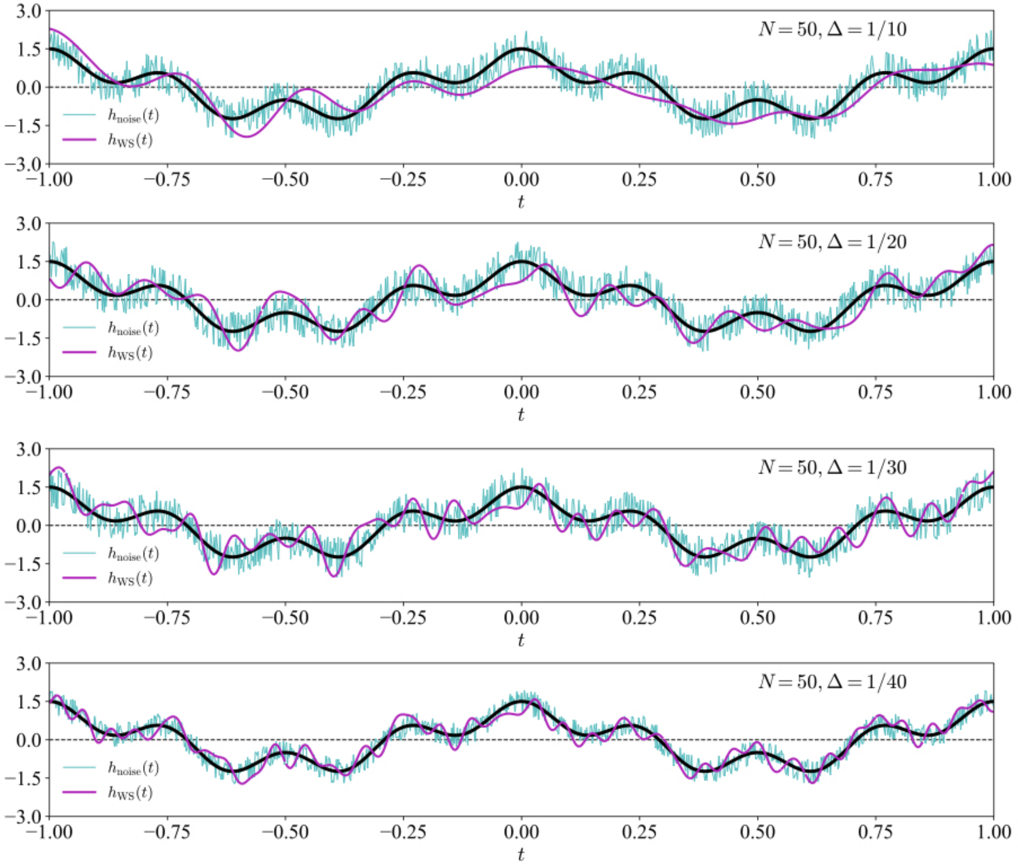
$$= \Delta \sum_{n=-\infty}^{n=\infty} 2f_c h_n \mathrm{sinc}[2\pi f_c(t-n\Delta)]$$



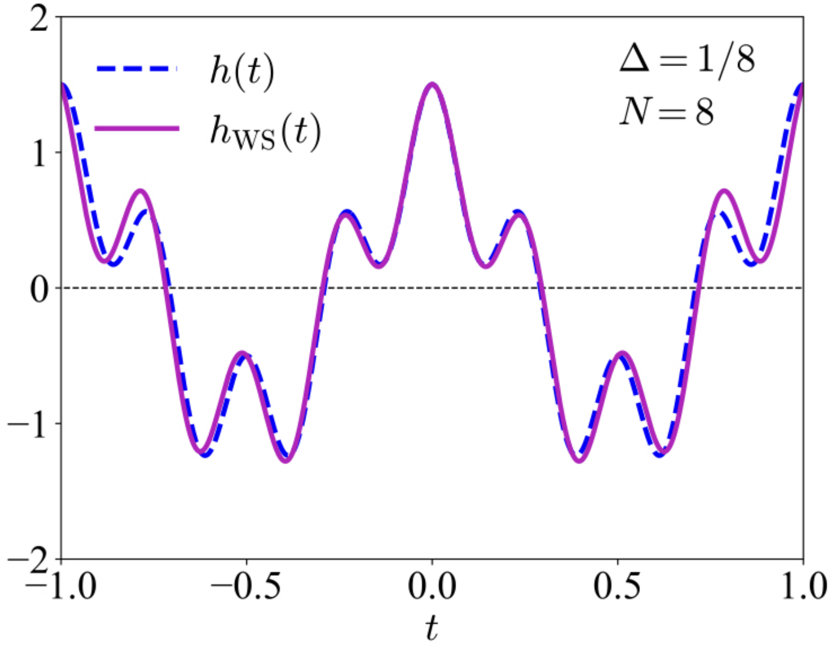


$$S(f) = \begin{cases} 1, & -f_0 \le f \le f_0, \\ 0, & \text{otherwise.} \end{cases}$$

Example of Whittaker-Shannon reconstruction



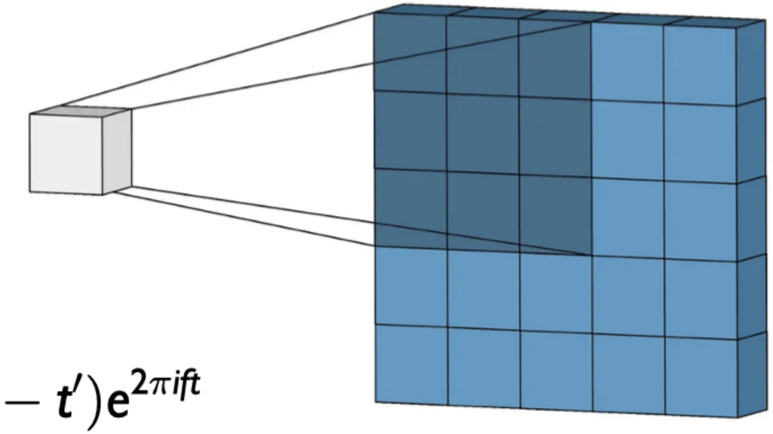
$$h_{\rm WS}(t) = \Delta \sum_{n=-\infty}^{\infty} h_n \frac{\sin[2\pi f_c(t-n\Delta)]}{\pi(t-n\Delta)}$$



$$h(t) = \cos 2\pi t + \frac{1}{2}\cos 8\pi t,\ \Delta^{-1} = \frac{1}{2f_{max}} = \frac{1}{8}$$

Convolution and correlation

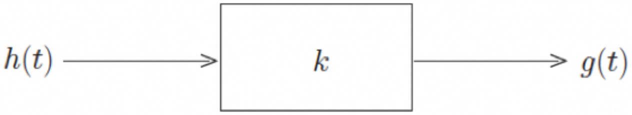
$$g(t) \equiv h(t) \star k(t) = \int dt' h(t') k(t-t')$$



convolution theorem

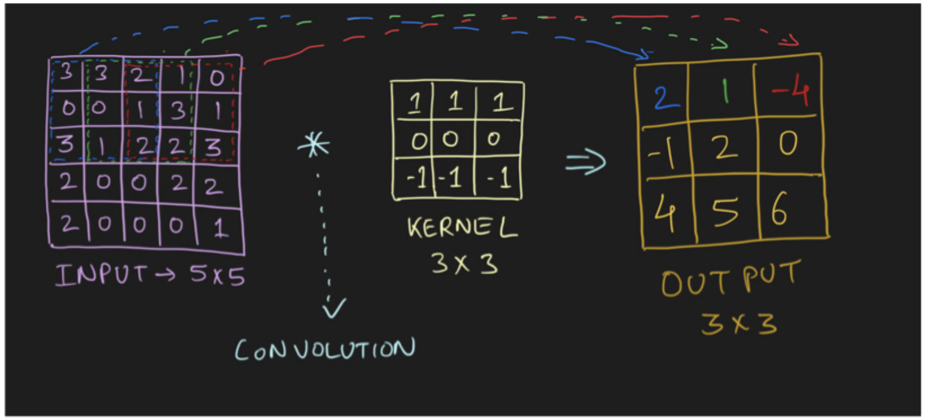
$$G(f) = \int_{-\infty}^{\infty} g(t) e^{2\pi i f t} dt = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt dt' h(t') k(t-t') e^{2\pi i f t}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt'' dt' h(t') k(t'') e^{2\pi i f (t'+t'')} = H(f) K(f), \; t - t' = t''$$



Ex.: correlation, $c(t) = \int_{-\infty}^{\infty} dt' h(t') g(t+t')$,

Show $C(f) = G(f)H(-f)$, what happens if g and h are real?



$$\begin{bmatrix} 3 & 3 & 2 & 1 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 3 & 1 & 2 & 2 & 3 \\ 2 & 0 & 0 & 2 & 2 \\ 2 & 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & -4 \\ -1 & 2 & 0 \\ 4 & 5 & 6 \end{bmatrix}$$

Discrete Fourier transform (DFT): notations

$$h_k \equiv h(t_k),\ t_k = k\Delta,\ f_n = \frac{n}{N\Delta},\ n = -\frac{N}{2}, \cdots, \frac{N}{2}$$

$$H(f_n) = \int_{-\infty}^{\infty} h(t) e^{2\pi i f_n t} dt \approx \sum_{k=1}^{N-1} h_k e^{2\pi i f_n t_k} \Delta = \Delta \sum_{k=0}^{N-1} h_k e^{2\pi i k n/N} = \Delta H_n$$

$$H_n \equiv \sum_{k=0}^{N-1} h_k e^{2\pi i k n/N},\ h_k = \frac{1}{N} \sum_{n=0}^{N-1} H_n e^{-2\pi i k n/N}$$

Discrete Fourier transform: divide-and-conquer

$$H_n \equiv \sum_{k=0}^{N-1} h_k e^{2\pi ikn/N} = \sum_{k=0}^{N-1} W^{nk} h_k, \;\; W = e^{2\pi i/N}$$

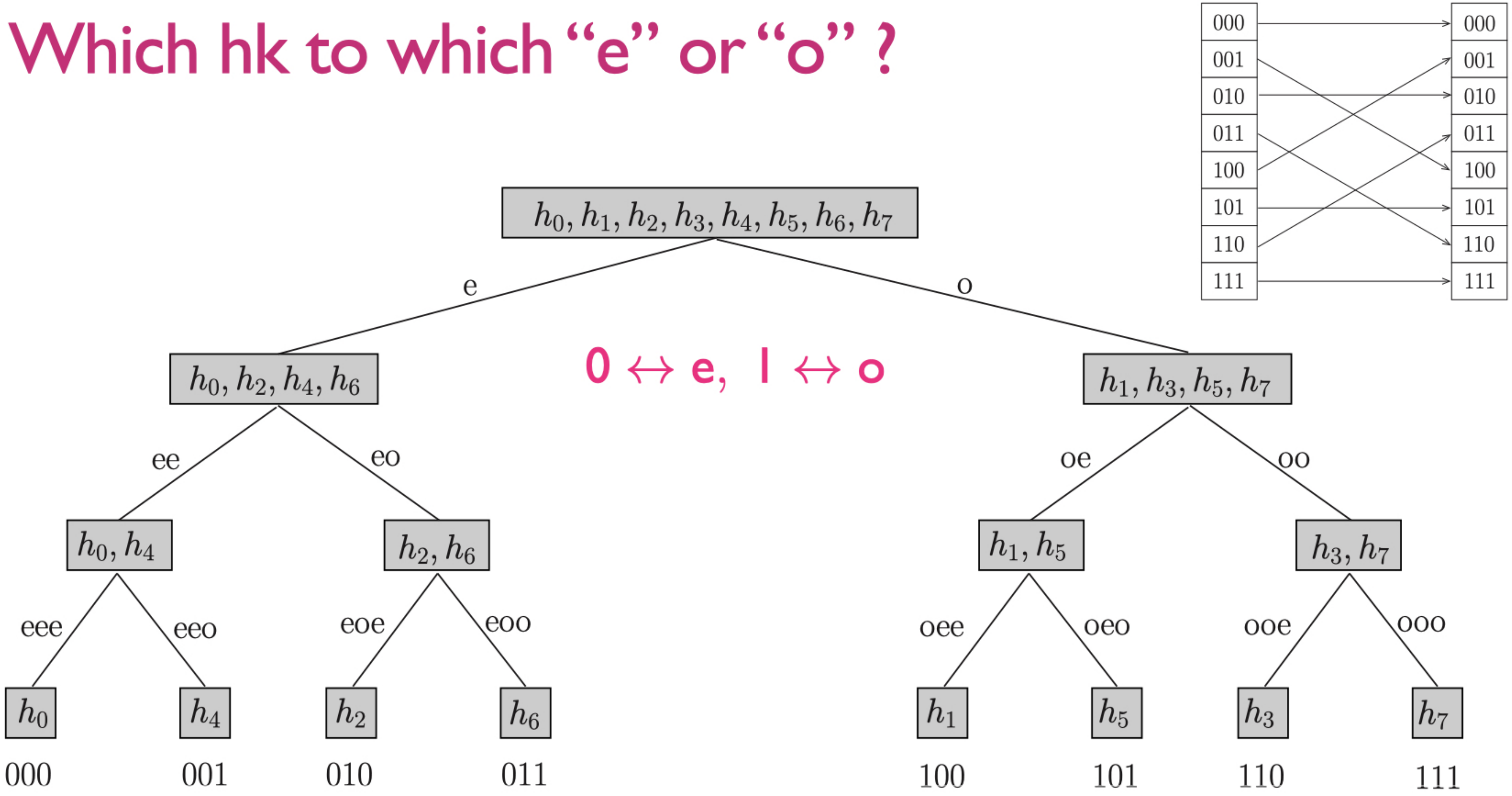
$$H_k^{\mathrm{eoeeeeo\cdots oee}} = h_n$$

$$H_k = \sum_{j=0}^{N-1} e^{2\pi ijk/N} h_j = \sum_{j=0}^{N/2-1} e^{2\pi ik(2j)/N} h_{2j} + \sum_{j=0}^{N/2-1} e^{2\pi ik(2j+1)/N} h_{2j+1}$$

$$= \sum_{j=0}^{N/2-1} e^{2\pi ikj/(N/2)} h_{2j} + W^k \sum_{j=0}^{N/2-1} e^{2\pi ikj/(N/2)} h_{2j+1} = H_k^{\mathrm{e}} + W^k H_k^{\mathrm{o}}$$

$$T(N) = 2T(N/2) + \mathcal{O}(N) \rightarrow T(N) \sim \mathcal{O}(N \log N)$$

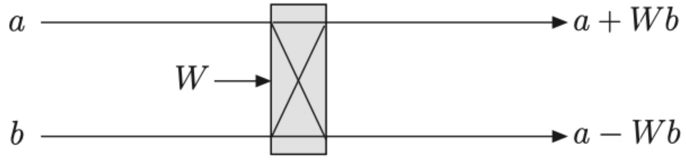
Which hk to which “e” or “o” ?

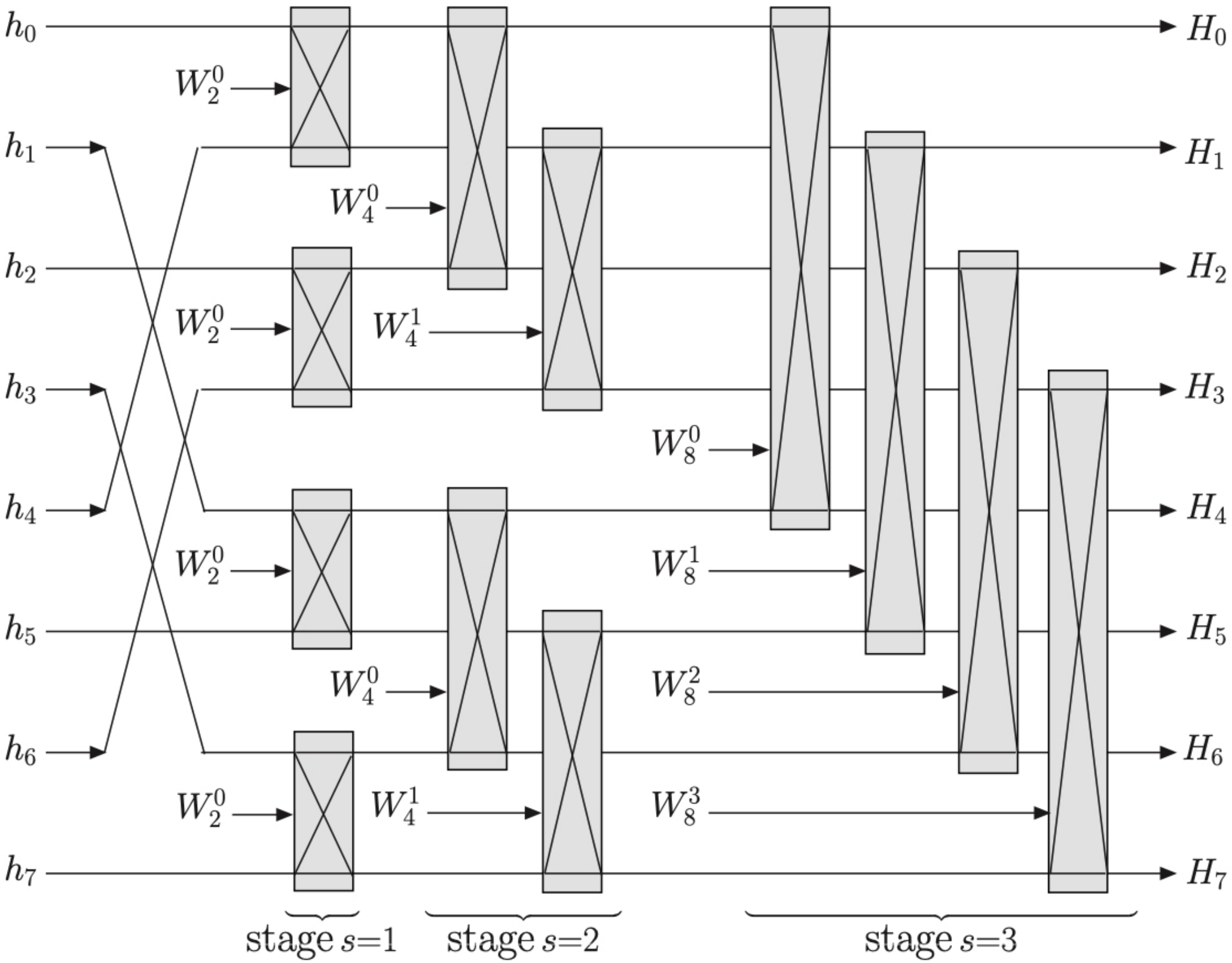


0 ↔ e, 1 ↔ o

FFT circuit

$$W_n = e^{2\pi i/n}$$





Algorithm for FFT (implementation)

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step (a)  Bit-reverse the input $\mathbf{h} = (h_0, h_1, \cdots, h_{N-1})$.
step (b)  For $s = 1 \ldots \log N$:
              $n \leftarrow 2^s, W_n \leftarrow e^{2\pi i/n}$.
              For $k = 0 \ldots N-1$ by $n$:
                  $\omega \leftarrow 1$.
                  For $j = 0 \ldots n/2-1$:
                      $u \leftarrow \omega h_{k+j+n/2}, v \leftarrow h_{k+j}$.
                      $h_{k+j} \leftarrow v + u, h_{k+j+n/2} \leftarrow v - u$.
                      $\omega \leftarrow \omega W_n$.
step (c)  Return $\mathbf{h}$.
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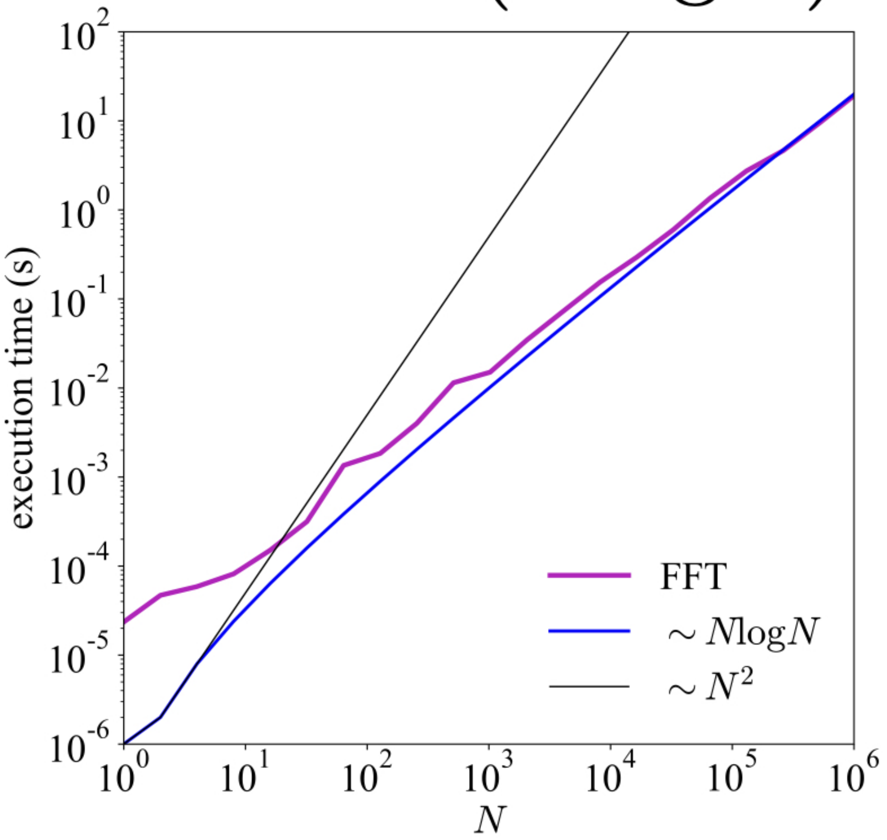
$$\sim \mathcal{O}(N \log N)$$



Ex.: Can you use FFT to multiply two polynomials?

Why FFT is so important? (from ChatGPT)

A Cornerstone of Modern Scientific Computing

- Converts complex signals and data into their frequency components.
- Enables efficient analysis of large-scale datasets.
- Forms the foundation of modern signal and image processing.
- Essential for communications technologies, including Wi-Fi, 5G, radar, and GPS.
- Widely used in physics, astronomy, fluid dynamics, quantum mechanics, and machine learning.
- Accelerates numerical simulations and the solution of differential equations.
- Makes real-time processing of audio, video, and sensor data possible.
- Regarded as one of the most influential algorithms of the 20th century.

James Cooley &John Tukey, 1965

Key Impact

FFT transformed Fourier analysis from a theoretical tool into a practical computational technology, enabling many of the scientific discoveries and digital technologies we rely on today.